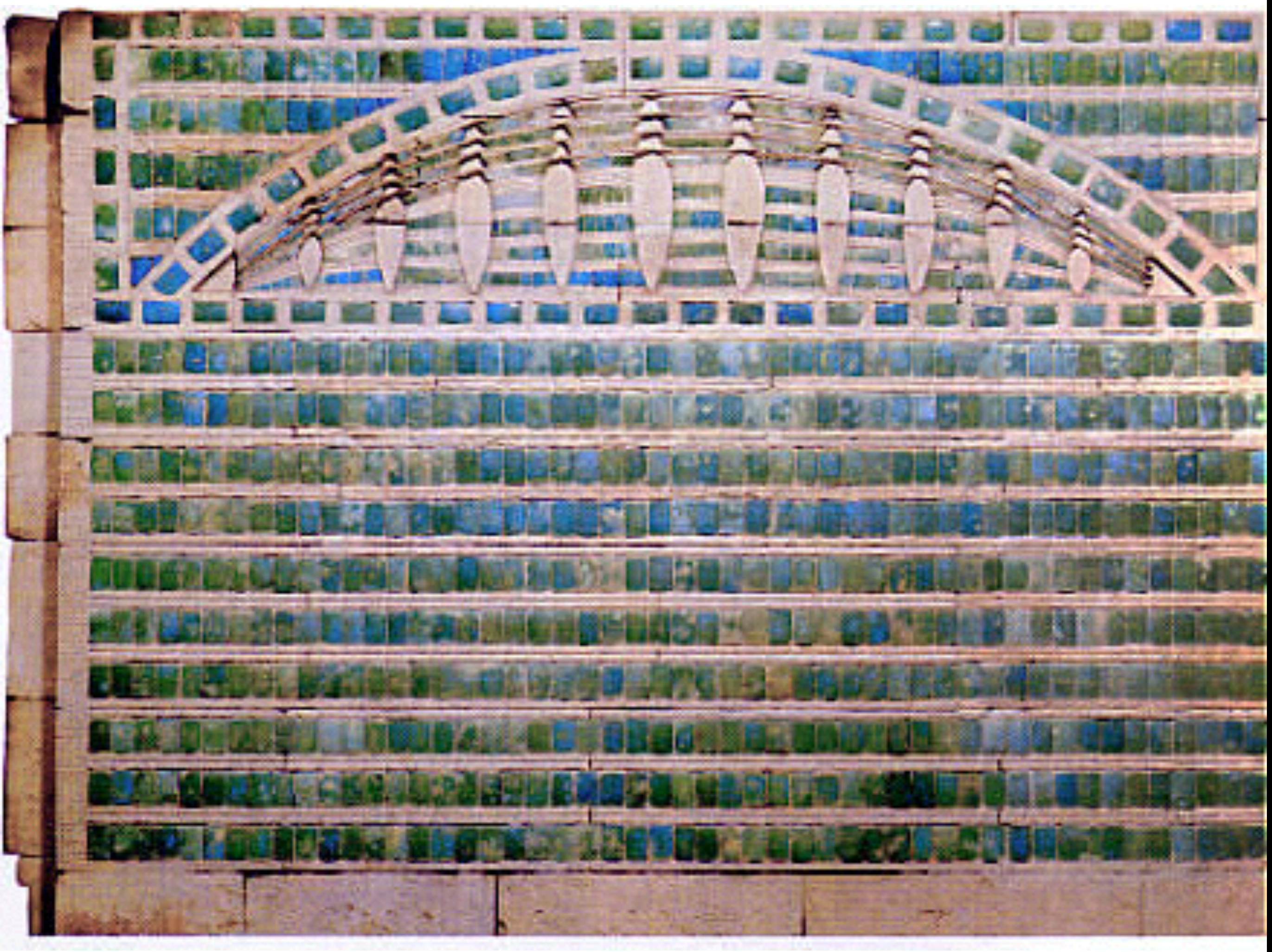
Tiling

Min Yan, Department of Mathematics

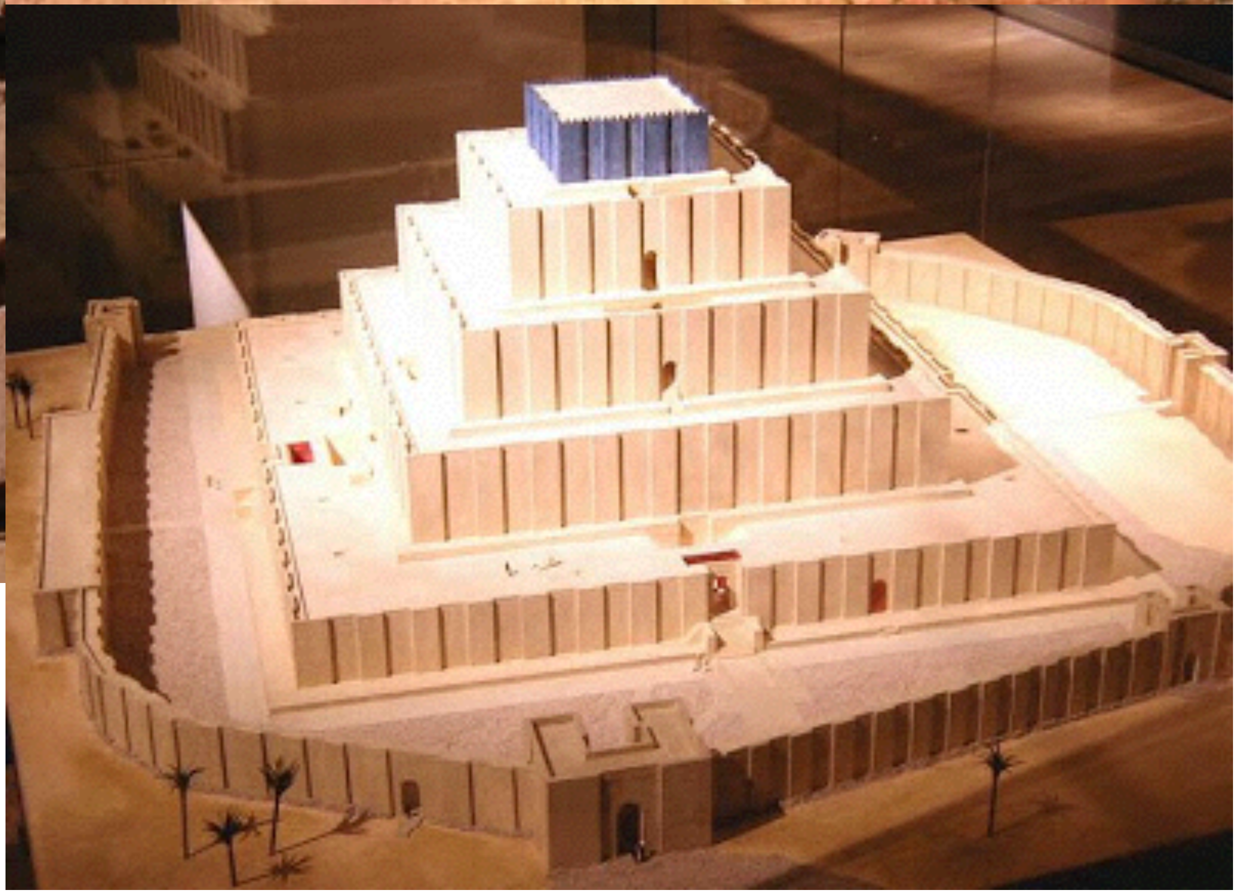
Uruk, Sumer (Iraq), 3400-3100BC

Tomb of King Djoser, Egypt, 2668-2649BC

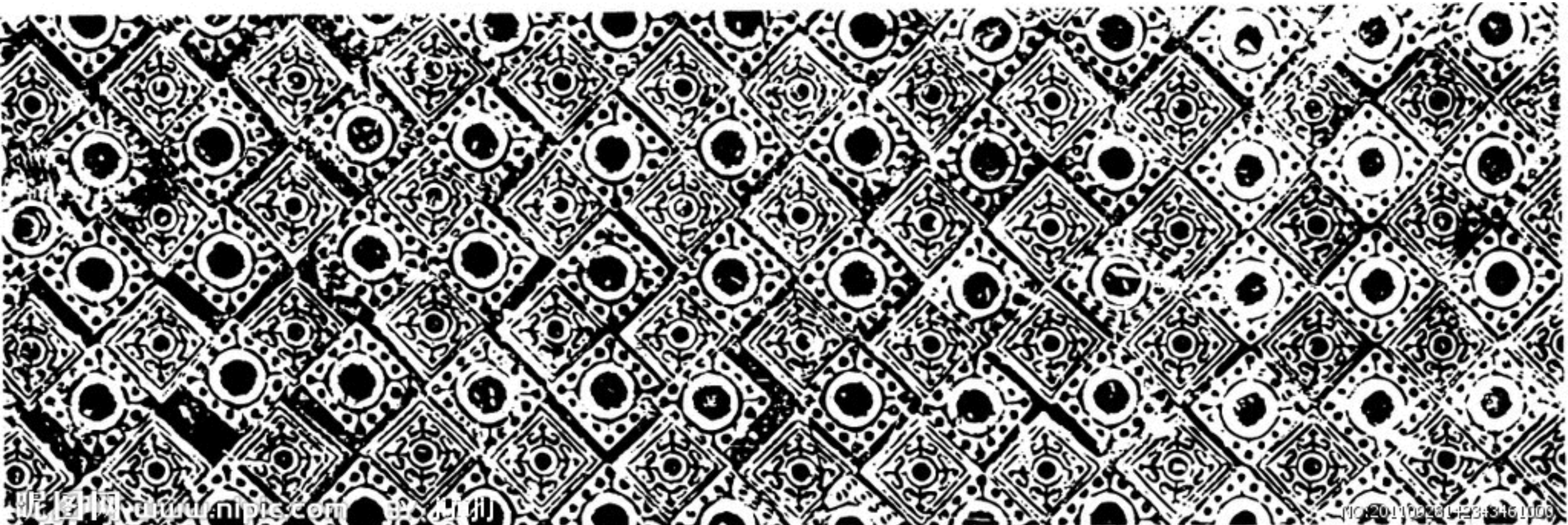


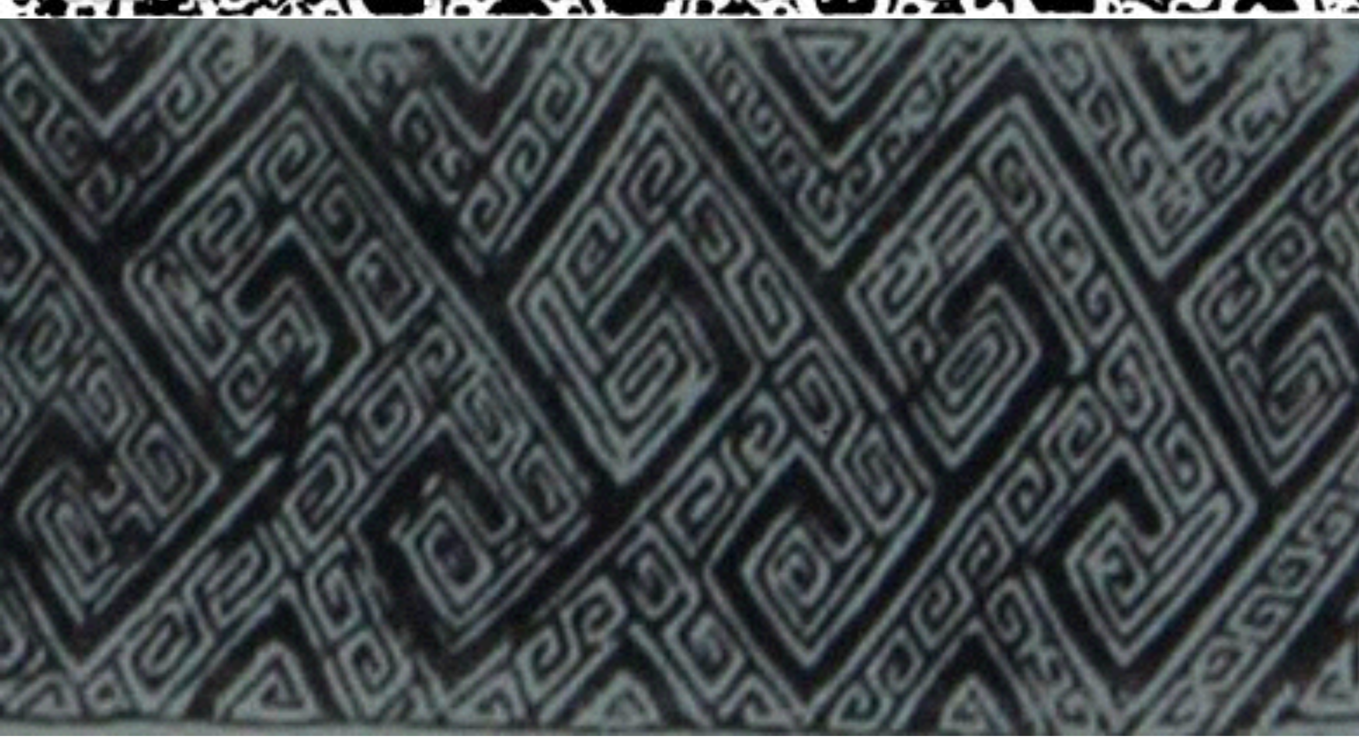
Chogha Zanbil, Elamite (Iran), 1250BC

Glazed brick tile



Chinese Bronze, ~1000BC





Ishtar Gate, Babylon (Iraq), 575BC

Pergaman Museum, Berlin

Greek

Delphi

Pella

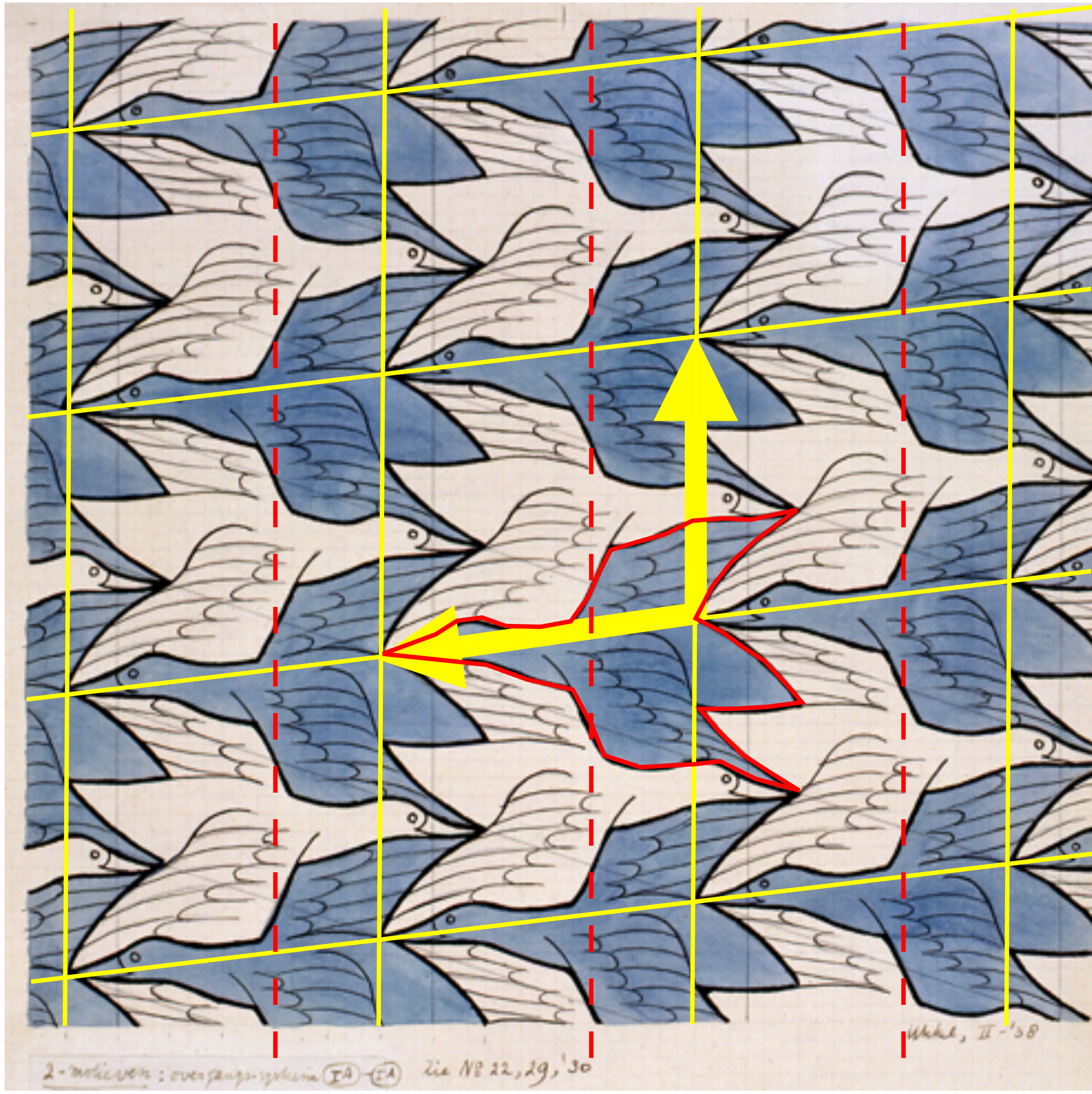
Roman



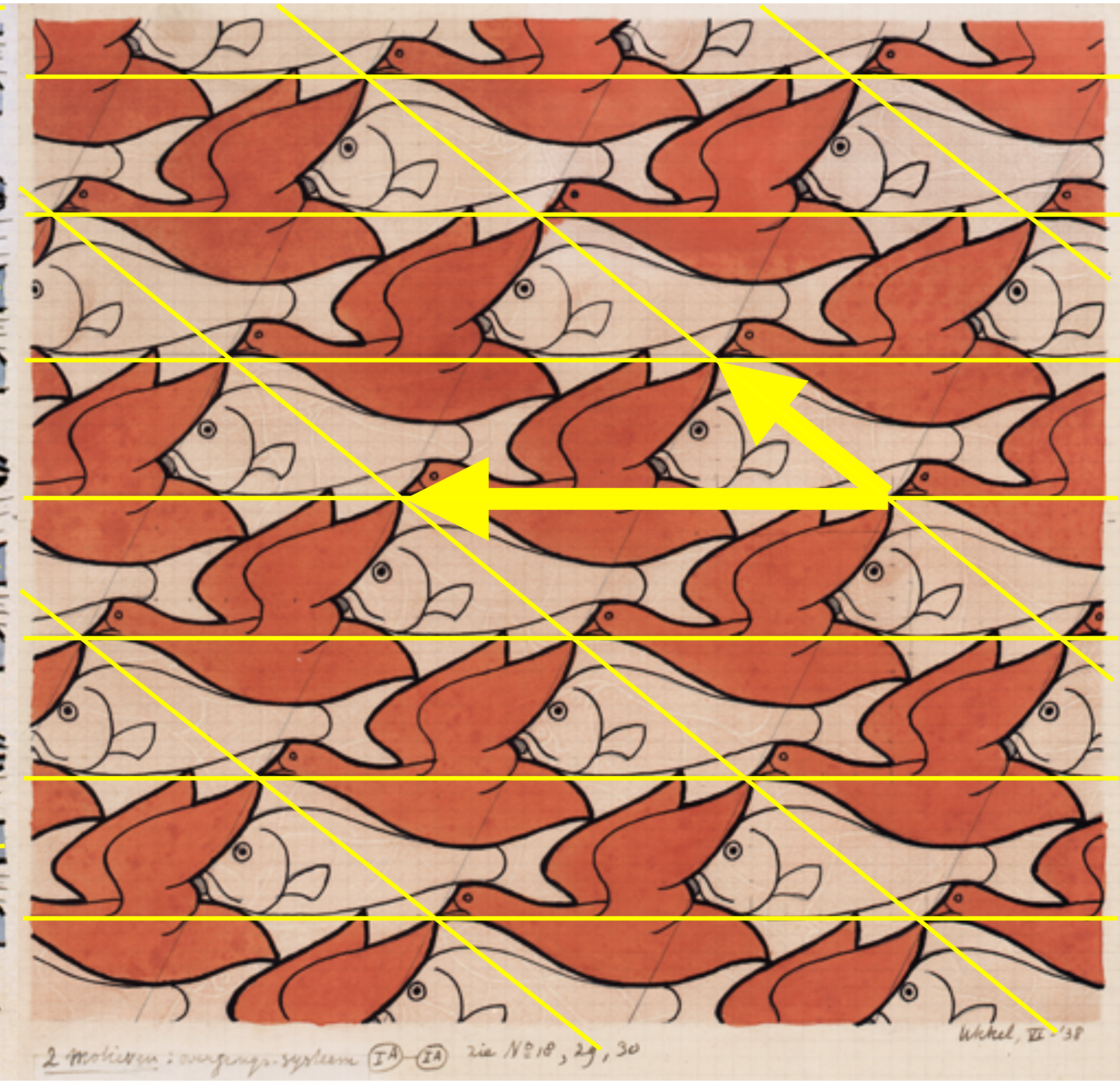

Dome of the Rock, Jerusalem (Israel), 16th c.

Islamic

M. C. Escher, 1898-1972



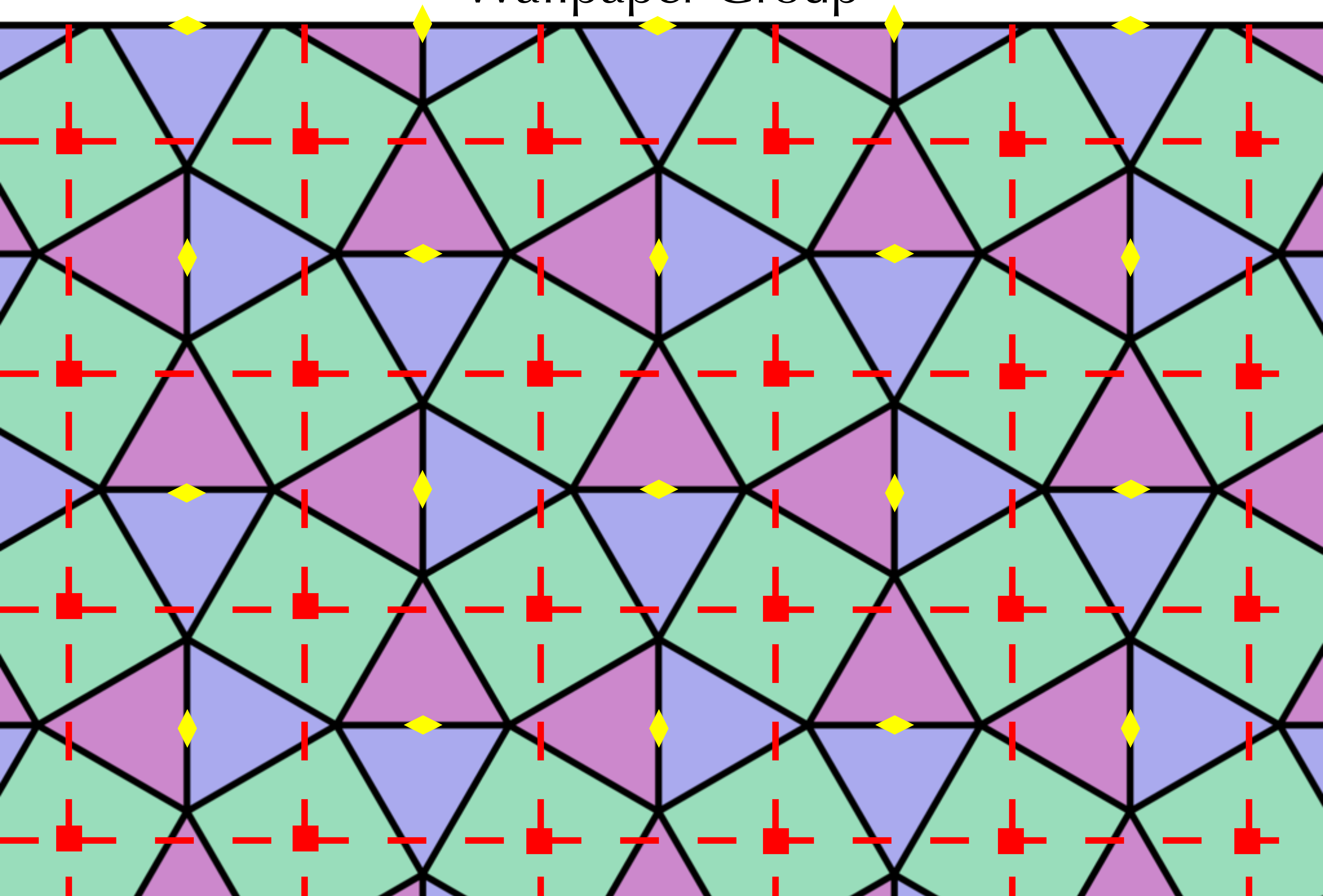
glide reflection



no glide reflection

Wallpaper Group

Wallpaper Group



Space Group

2D: 17 wallpaper groups

translation, rotation, reflection, glide reflection

known for centuries, proved in 1891 by Evgrad Fedorov

3D: 230 crystallographic groups

fixing point: 32 point groups, consisting of rotation, reflection, improper rotation

translations: 14 types of Bravais lattice

glide: 5 glides with respect to a plane, 1 screw axis with respect to a line

1879 Leonhard Sohncke listed 65 space groups

1891 Evgrad Fedorov proved the complete list

number of space groups: (1D)2, (2D)17, (3D)230, (4D)4894, (5D)222097, (6D)28934974, …

1911 Bieberbach Theorem

1978 4D by Neubüser

2000 5D and 6D Plesken and Schulz

Tiling of Plane by Regular Polygons

Johannes Kepler (1571-1630)

Mathematician

Astronomer

Astrologer

Johannes Kepler (1571-1630)

1596 Mysteriu)

1604 Astronor

1609 Astronor

1610 Disserta

1610 Narratio

1611 Dioptric lescope)

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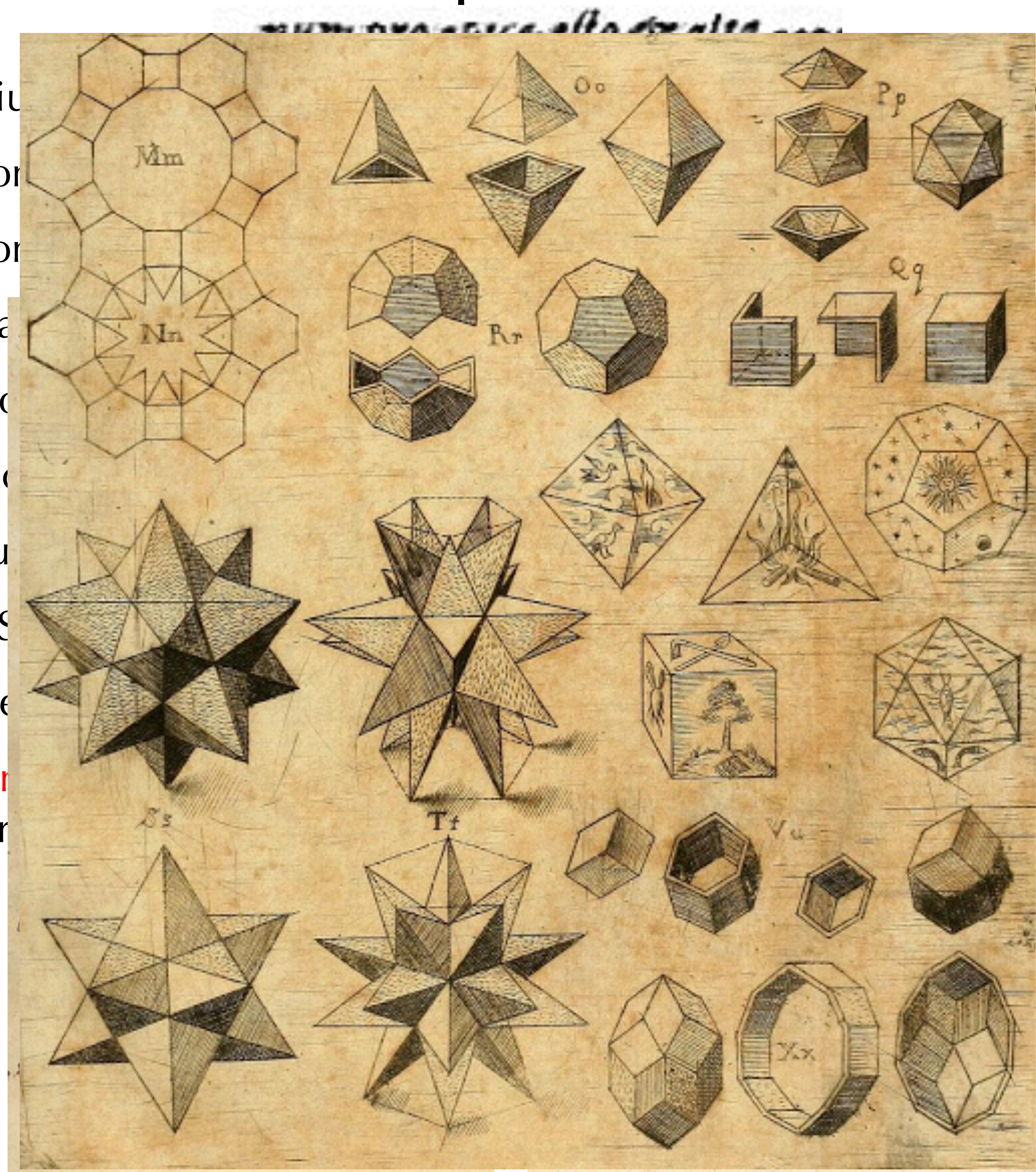
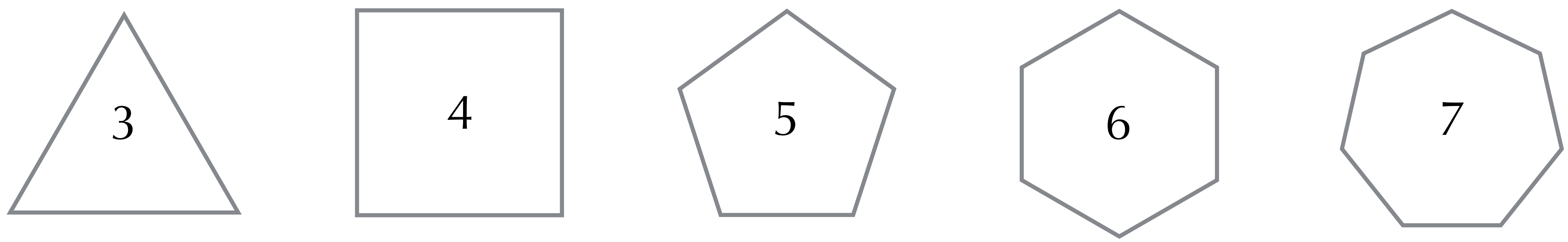
1615 Epitome w)

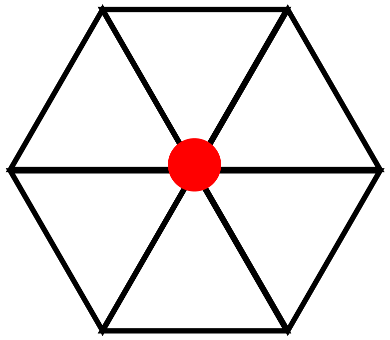
1619 Harmor eator with
the model for



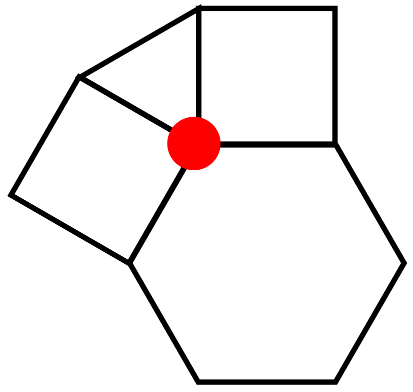
Regular Polygons



At a vertex, we need to fill 2π: $(n_1-2)/n_1 + (n_2-2)/n_2 + \ldots = 2 \Rightarrow$ vertex type

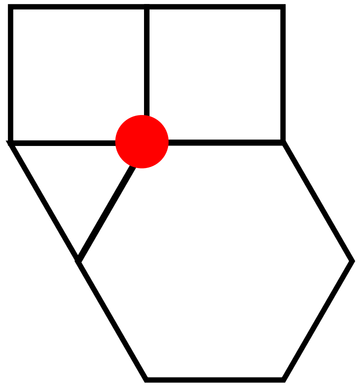


$1/3+1/3+1/3+1/3+1/3+1/3 = 2 \Rightarrow$ 3.3.3.3.3.3



$1/3+2/4+4/6+2/4 = 2 \Rightarrow$ 3.4.6.4

3.4.4.6 $\Leftarrow 1/3+2/4+2/4+4/6 = 2$



All Vertex Types

most perfect

3.3.3.3.3.3, 4.4.4.4, 6.6.6

perfect

3.3.3.4.4, 3.3.4.3.4, 3.3.3.3.6, 3.6.3.6, 3.12.12, 4.6.12, 3.4.6.4, 4.8.8

imperfect

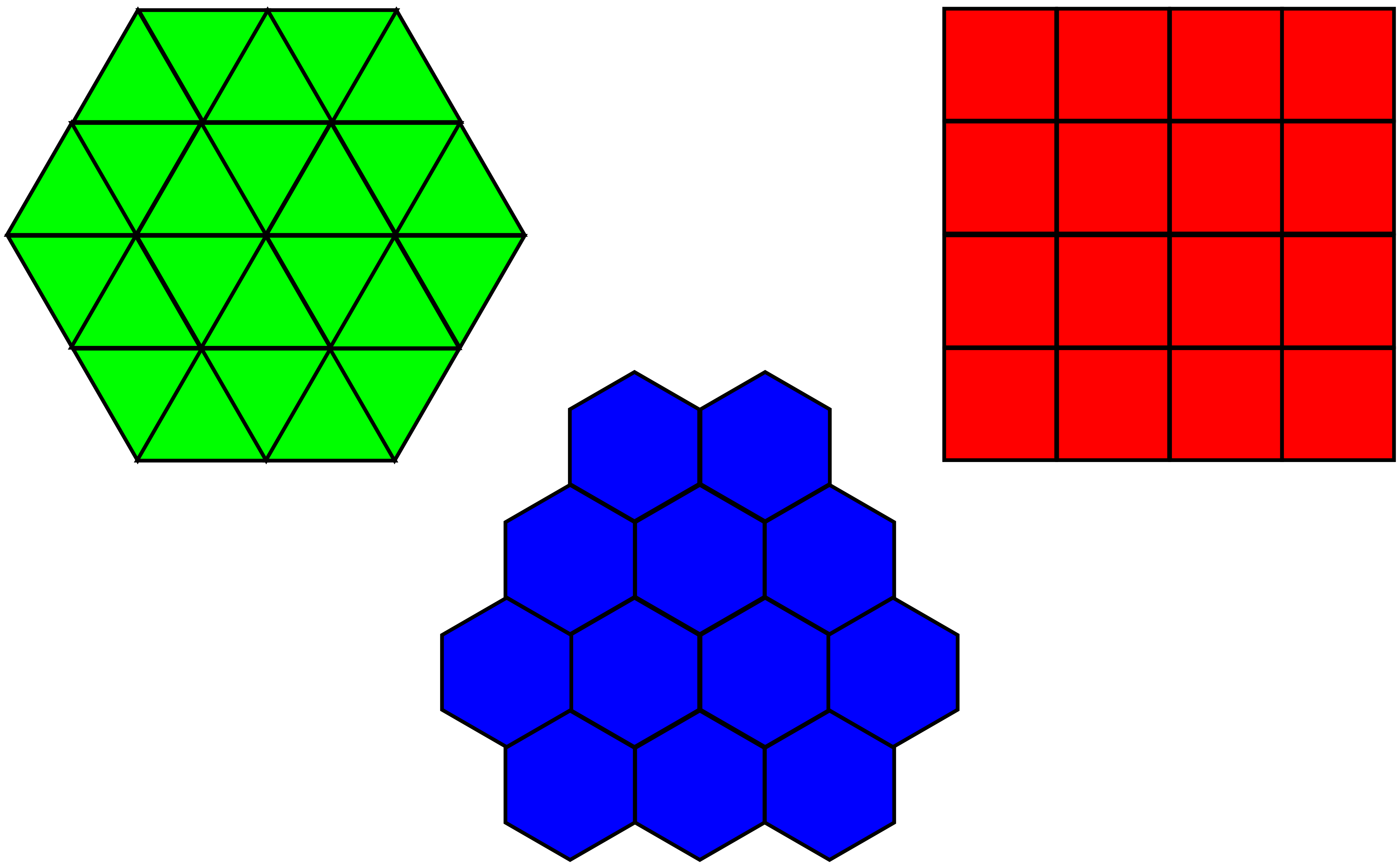
3.3.4.12, 3.4.3.12, 3.3.6.6, 3.4.4.6

impossible

3.7.42, 3.8.24, 3.9.18, 3.10.15, 4.5.20, 5.5.10

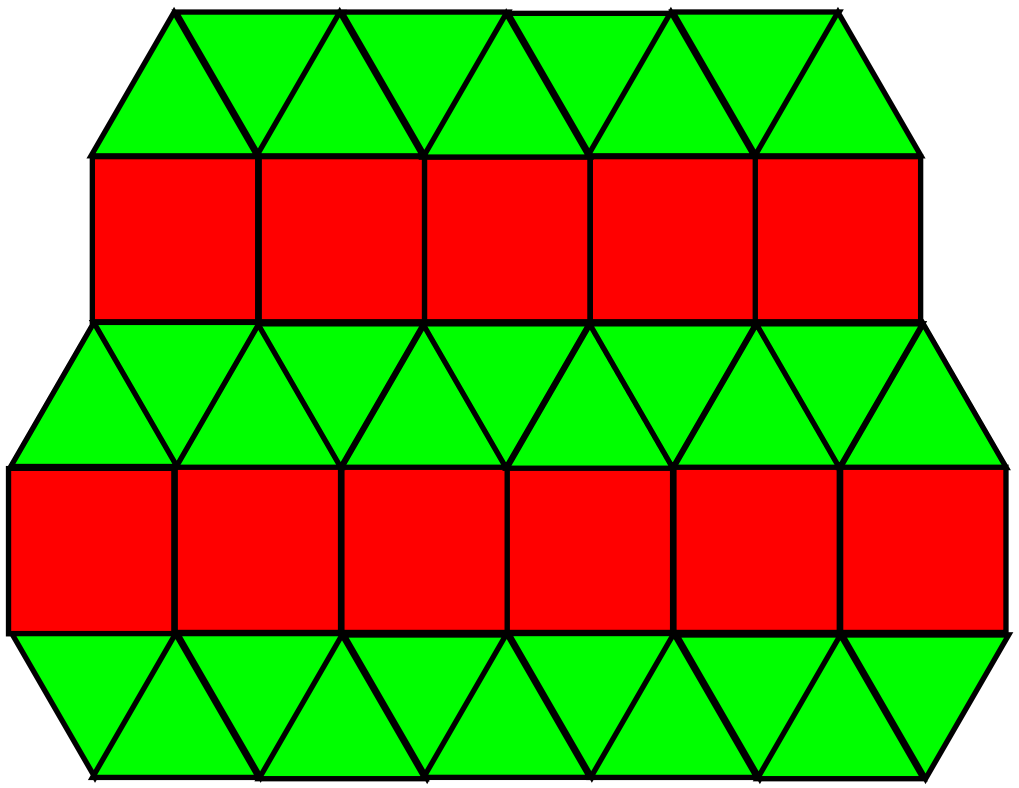
Most Perfect

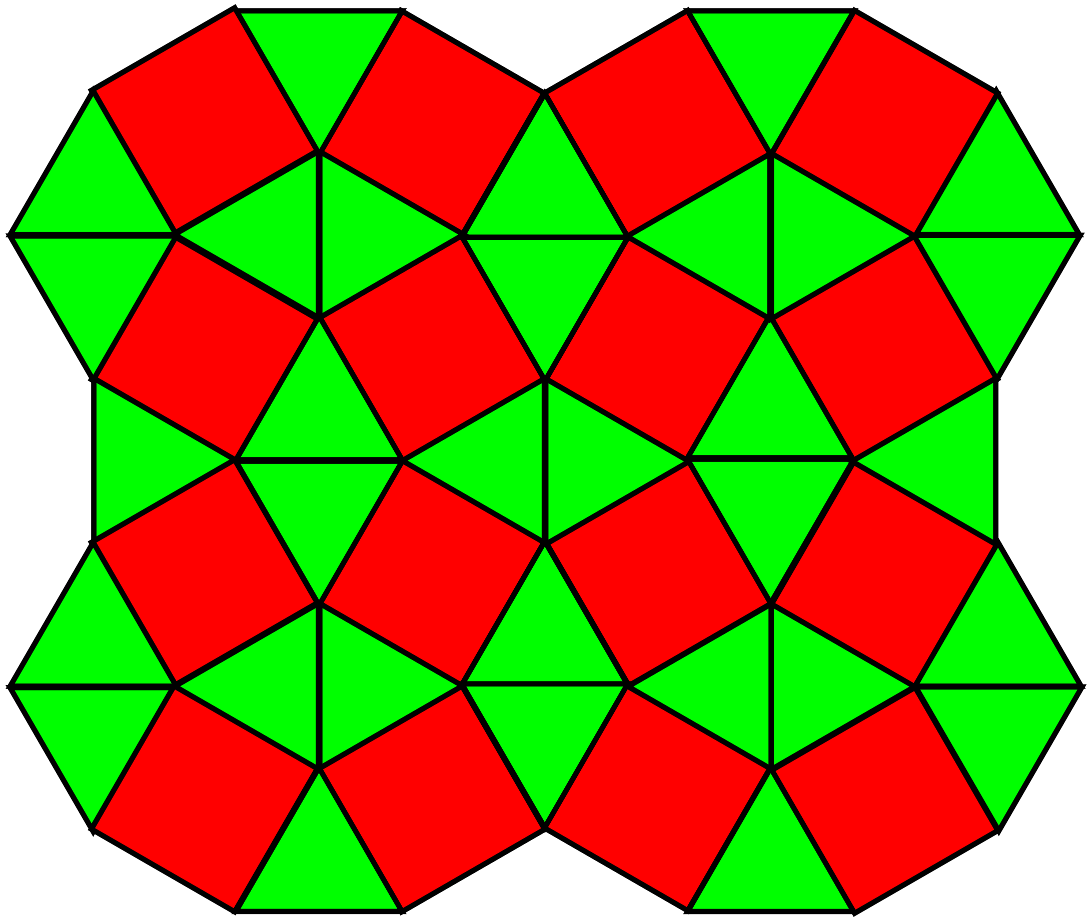
only one gon



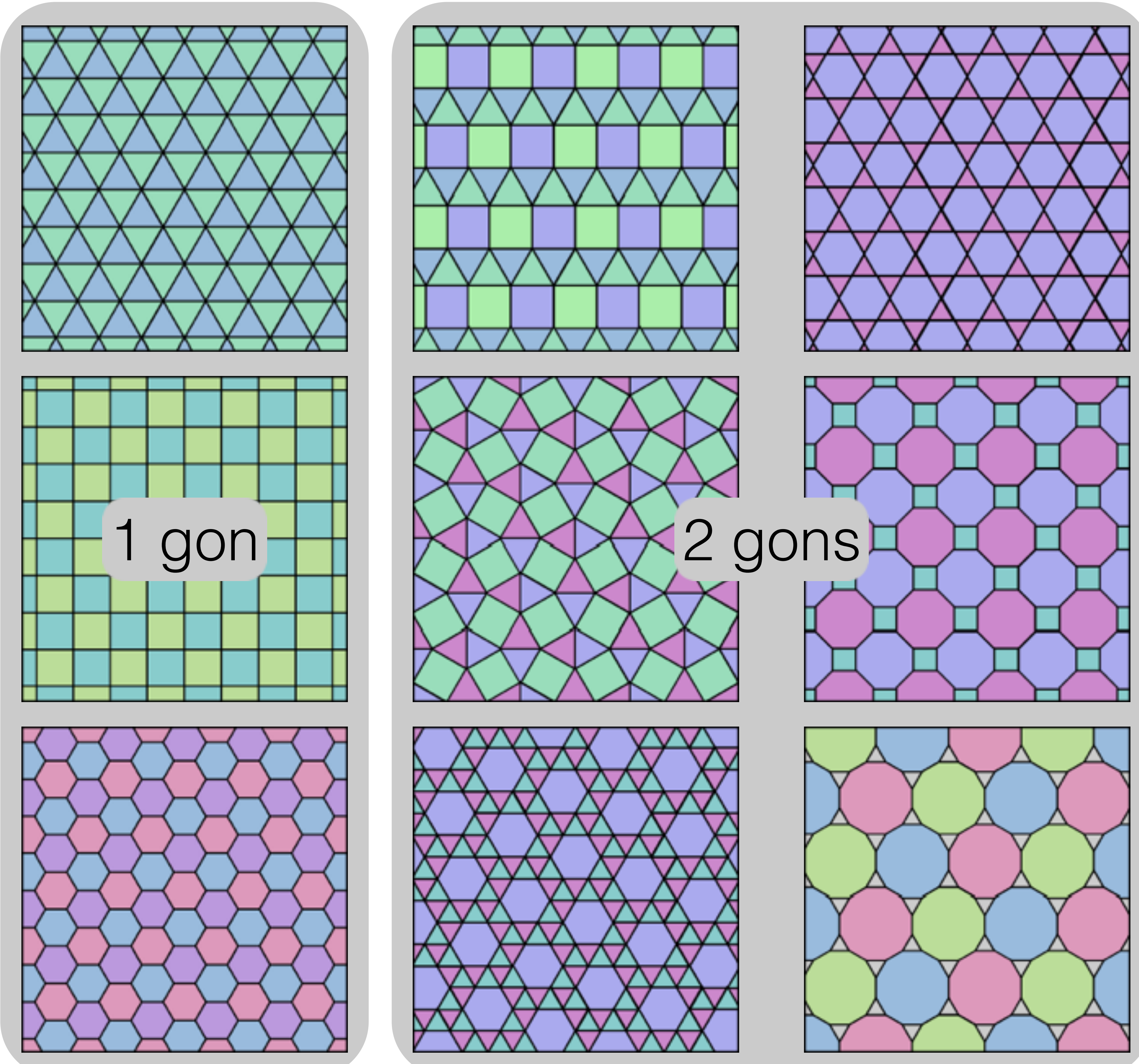
Perfect

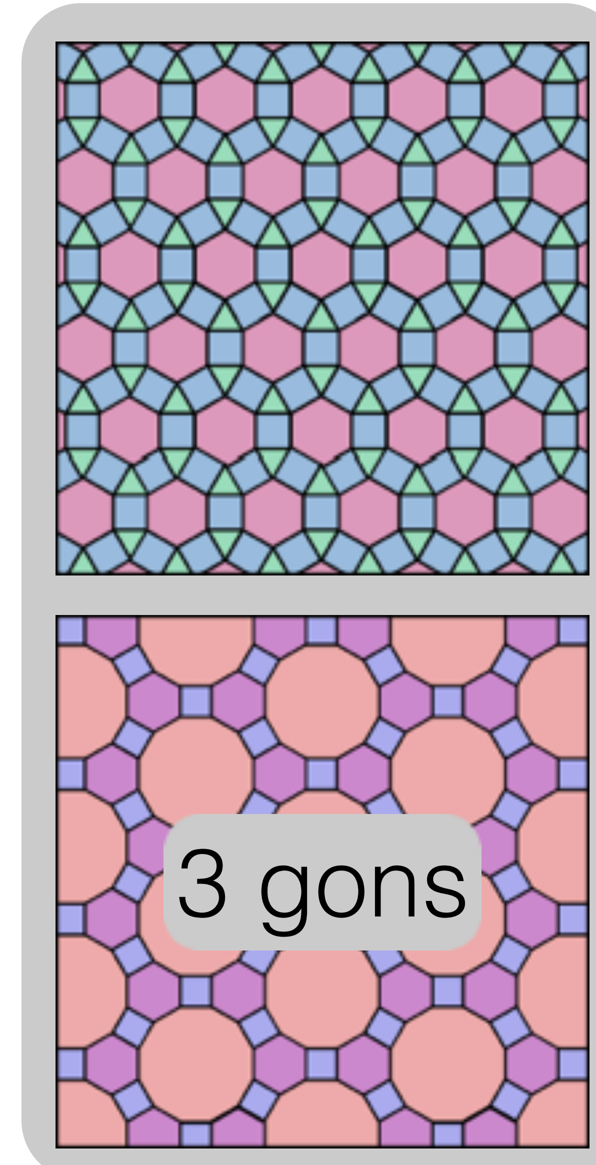
two or more gons, but only one vertex type





Archimedean





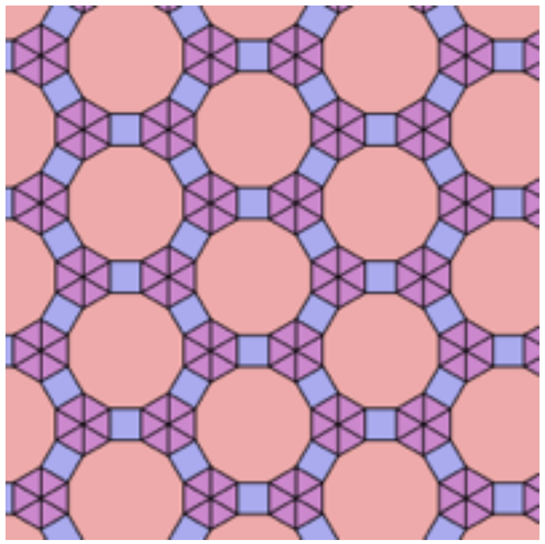
only one vertex type

⇳

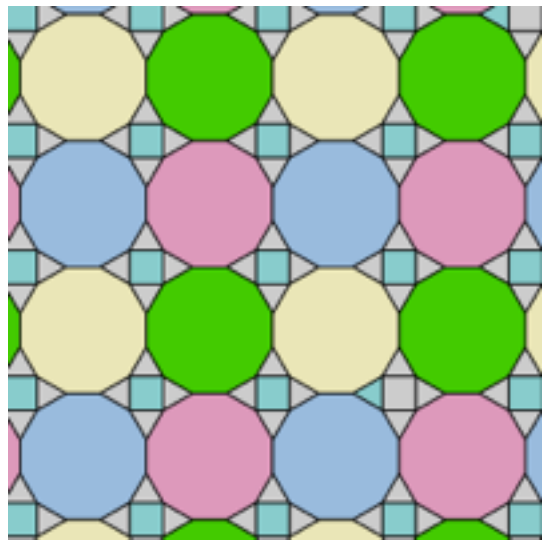
most perfect + perfect

Imperfect

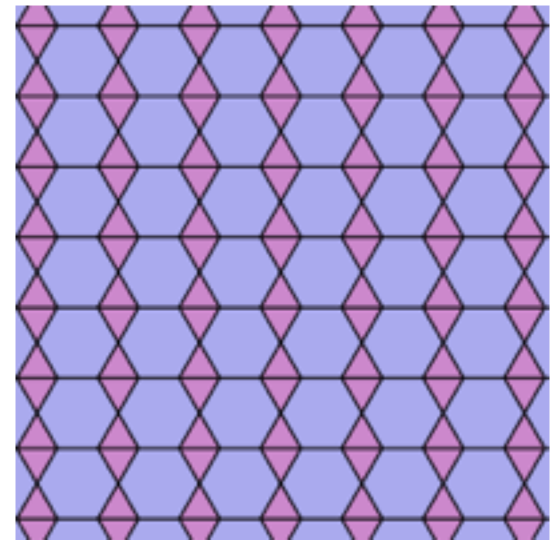
two or more vertex types



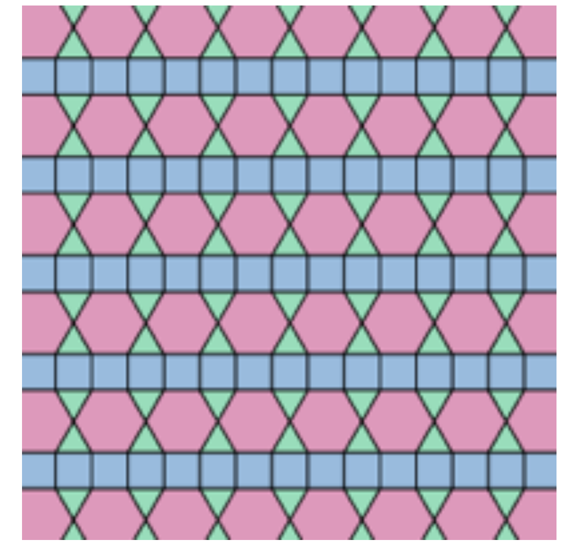
3.3.4.12
3.3.3.3.3.3



3.4.3.12
3.12.12



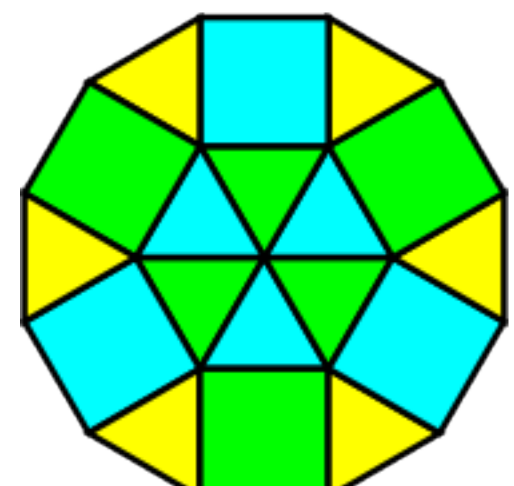
3.3.6.6
3.6.3.6



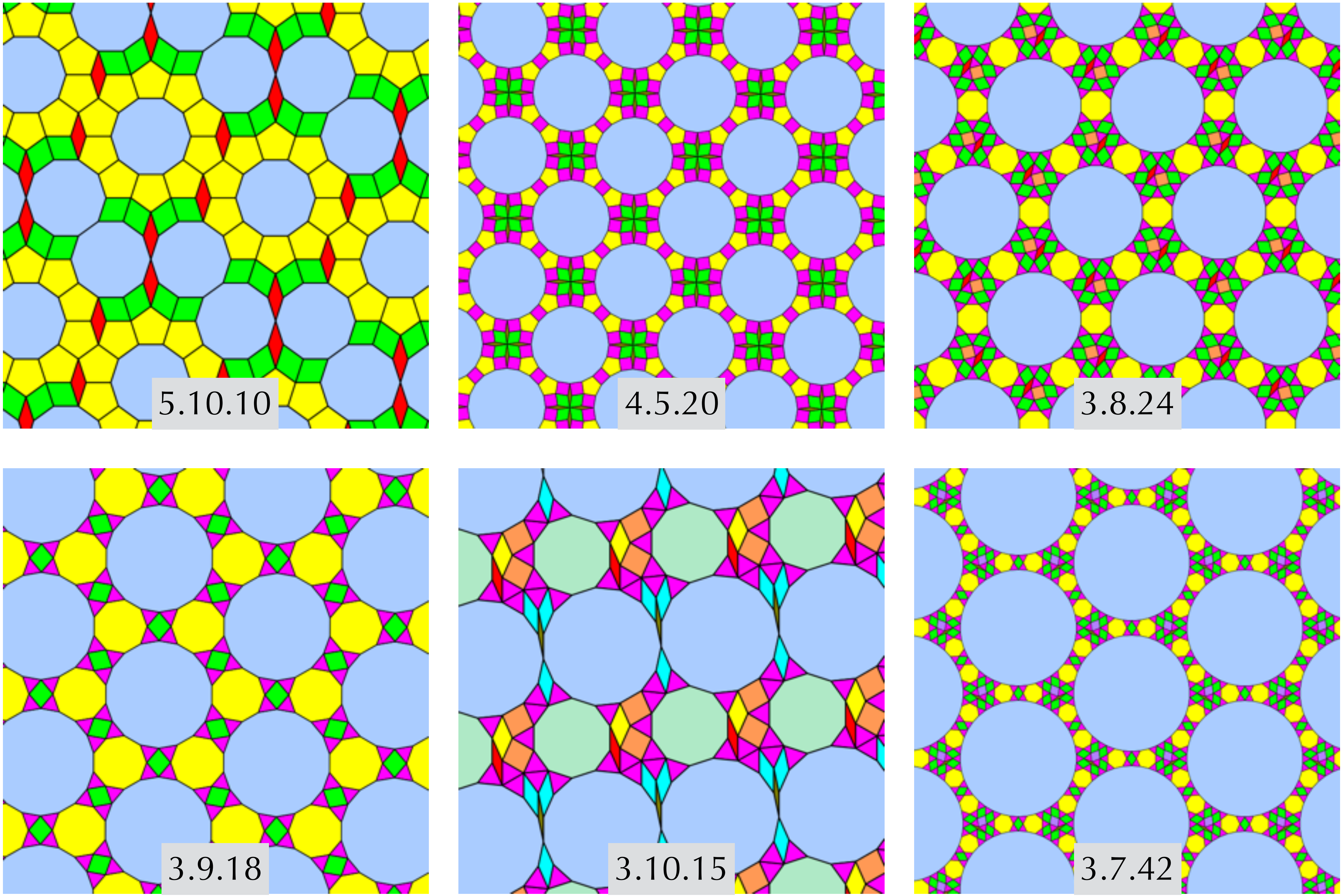
3.4.4.6
3.6.3.6

Many more examples …

And the 12-gon is a combination of 3-gons and 4-gons



Tiling the Impossibles by Adding Rhombus



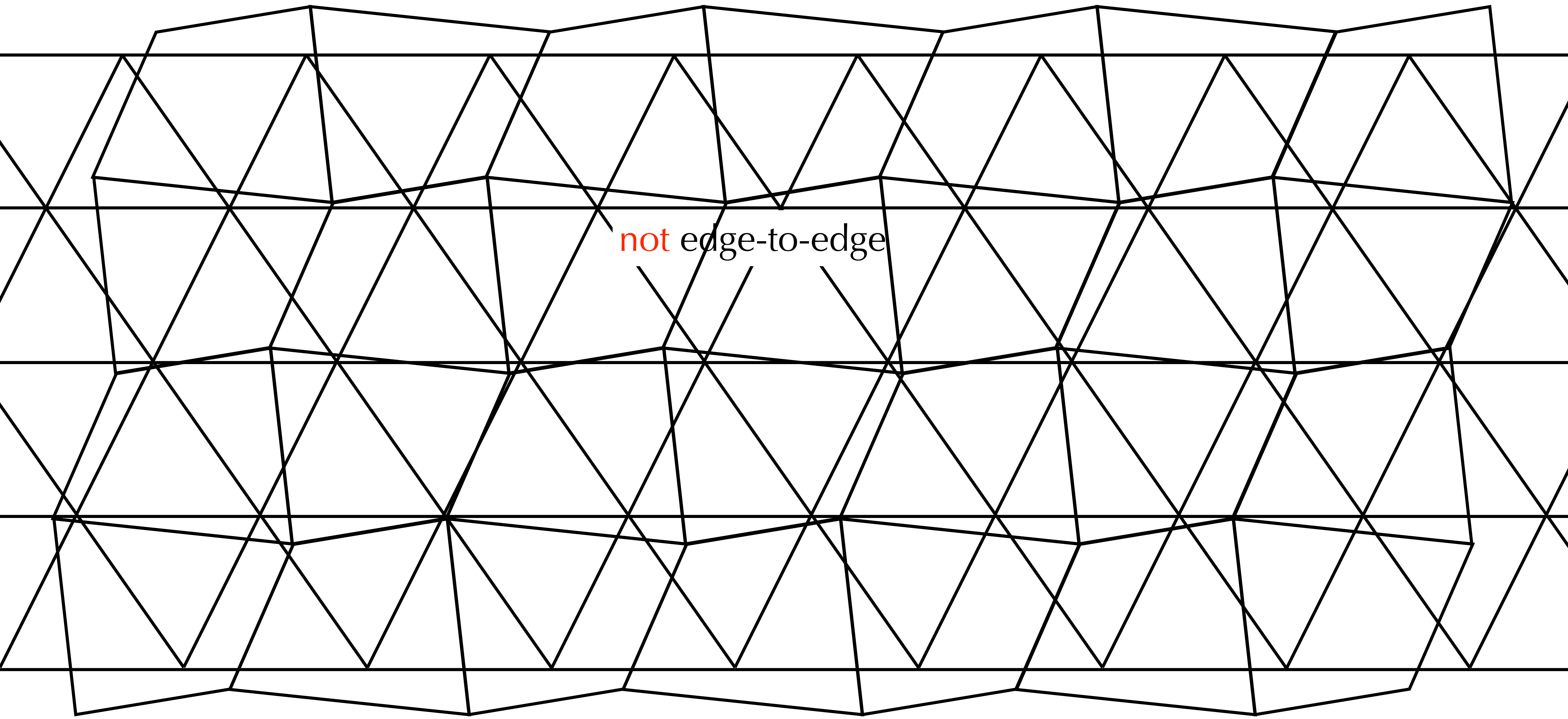
Convex Monohedral Tile

Convex Monohedral Tile

Use one (= monohedral) convex (not necessarily regular) polygon to tile the plane

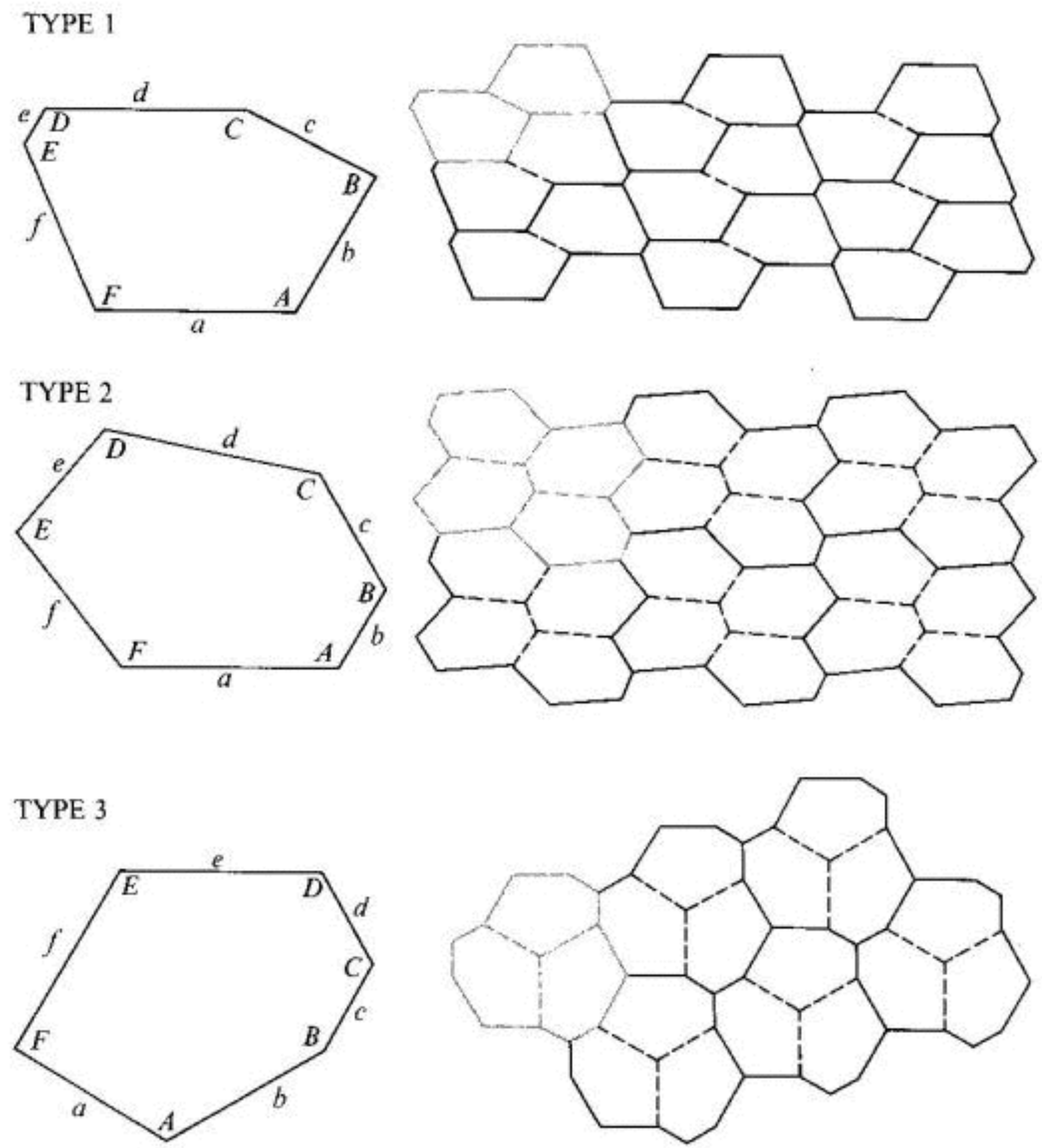
Any triangle tiles the plane

Any quadrilateral tiles the plane

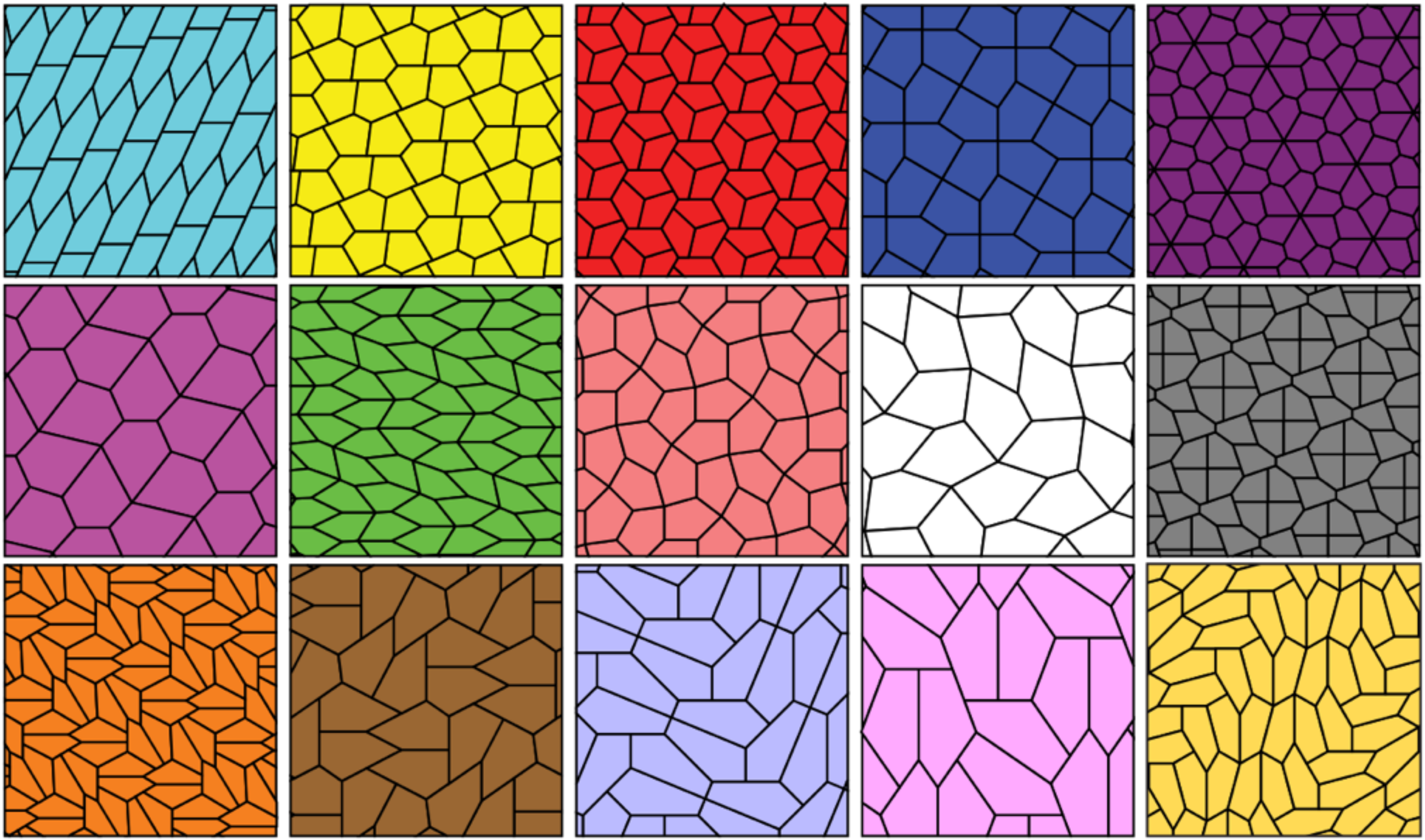


3 Convex Hexagon Tiles

1918, K. Reinhardt



15 Known Convex Pentagon Tiles



14 Known Convex Pentagon Tiles

1918 K. Reinhardt, types 1-5, transitive

1968 R. B. Kershner, types 6-8

1975 R. James, type 10

1976-1977 M. Rice, types 9, 11-13

1985 R Stein, type 14

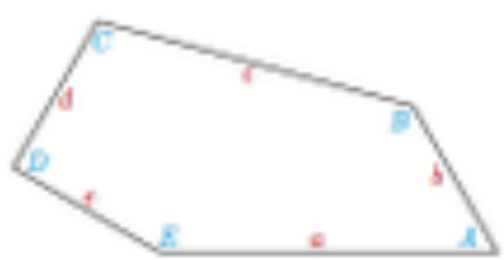
2015 J. McLoud-Mann, C. Mann, D. V. Derau, type 15

O. Bagina (2011) and T. Sugimoto (2012) proved 8 edge-to-edge convex types.

Type 15 Convex Pentagon

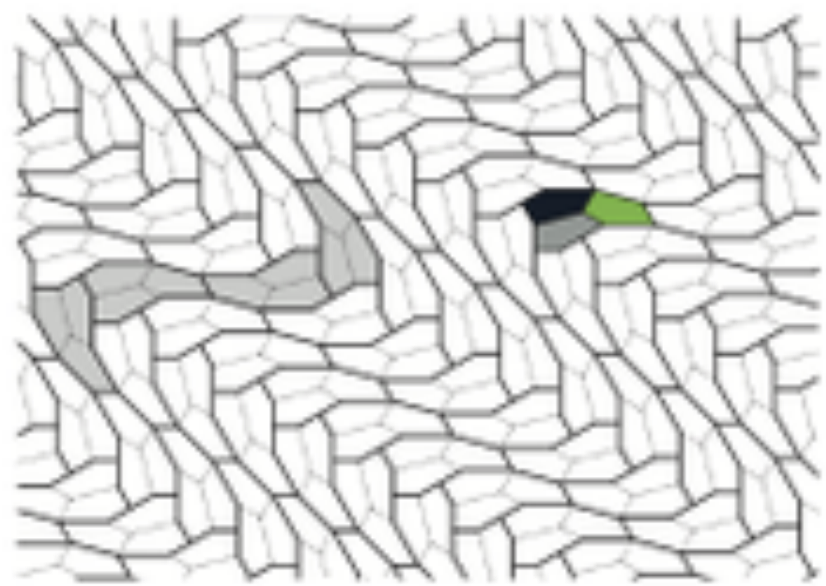
Casey Mann, University of Washington Bothell Jennifer McLoud, University of Washington Bothell
David Von Derau, University of Washington Bothell

July 29, 2015



(a)

$A = 60^\circ$ $a = 1$
$B = 135^\circ$ $b = 1/2$
$C = 105^\circ$ $c = \dfrac{1}{\sqrt{2}(\sqrt{3}-1)}$
$D = 90^\circ$ $d = 1/2$
$E = 150^\circ$ $e = 1/2$



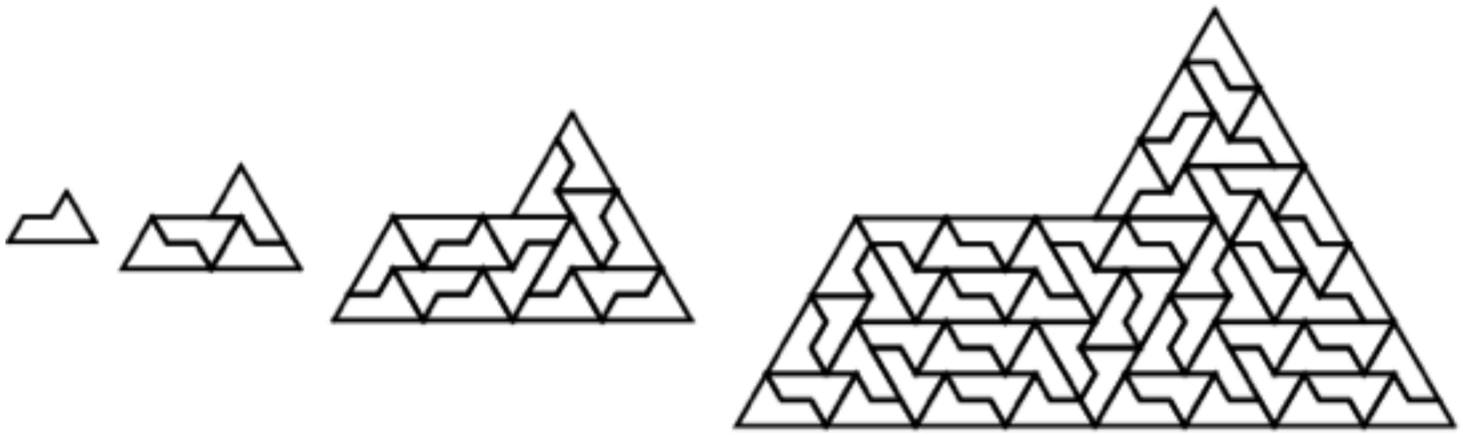
(b)
The grey patch is a fundamental region for the tiling. The tiling is tile-3-transitive. The 3-blocks for the tiling are outlined in heavy lines. Representatives of the 3 transitivity classes (with repect to the symmetry group of the tiling) are colored blue, green, and grey at right.

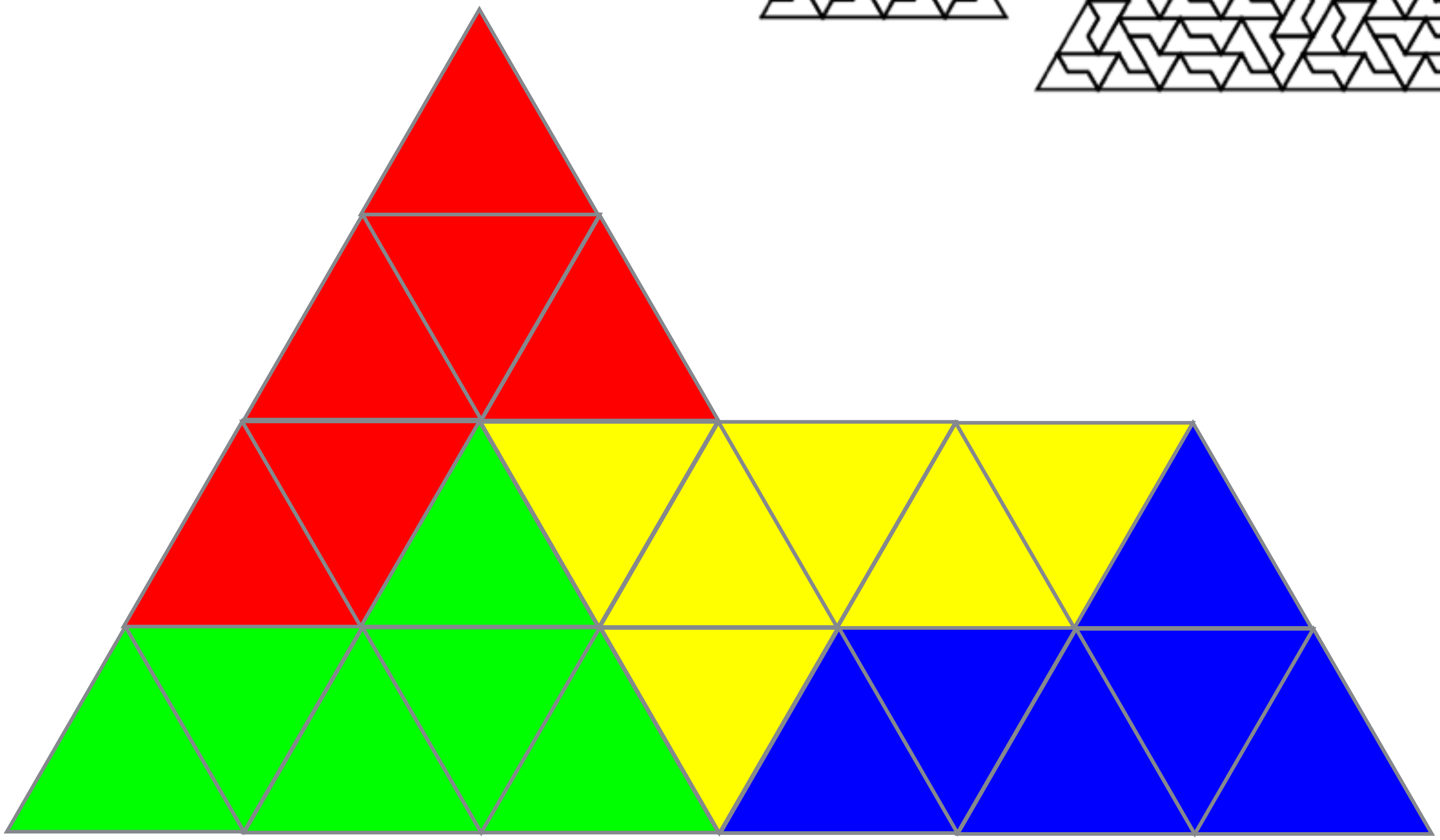
The pentagon of Figure 1a admits a periodic tiling of the plane as pictured in 1b. This tiling is tile-3-transitive. The corresponding tiling by 3-blocks (the patches outlined in dark lines) has isohedral type IH5. It is quickly determined that this tile is not among the known 14 types seen here: `http://www.mathpuzzle.com/tilepent.html`. Notice that in the tile of Figure 1a, there are no supplementary pairs of angles, so that rules out types 1, 2, 4, 6, 10, 11, 12, 13, and 14. Also, there are no 120° angles, so that rules out types 3 and 5. Since there are not 4 equal sides, that rules out types 7, 8, and 9.

Rep-tile

Rep-tile

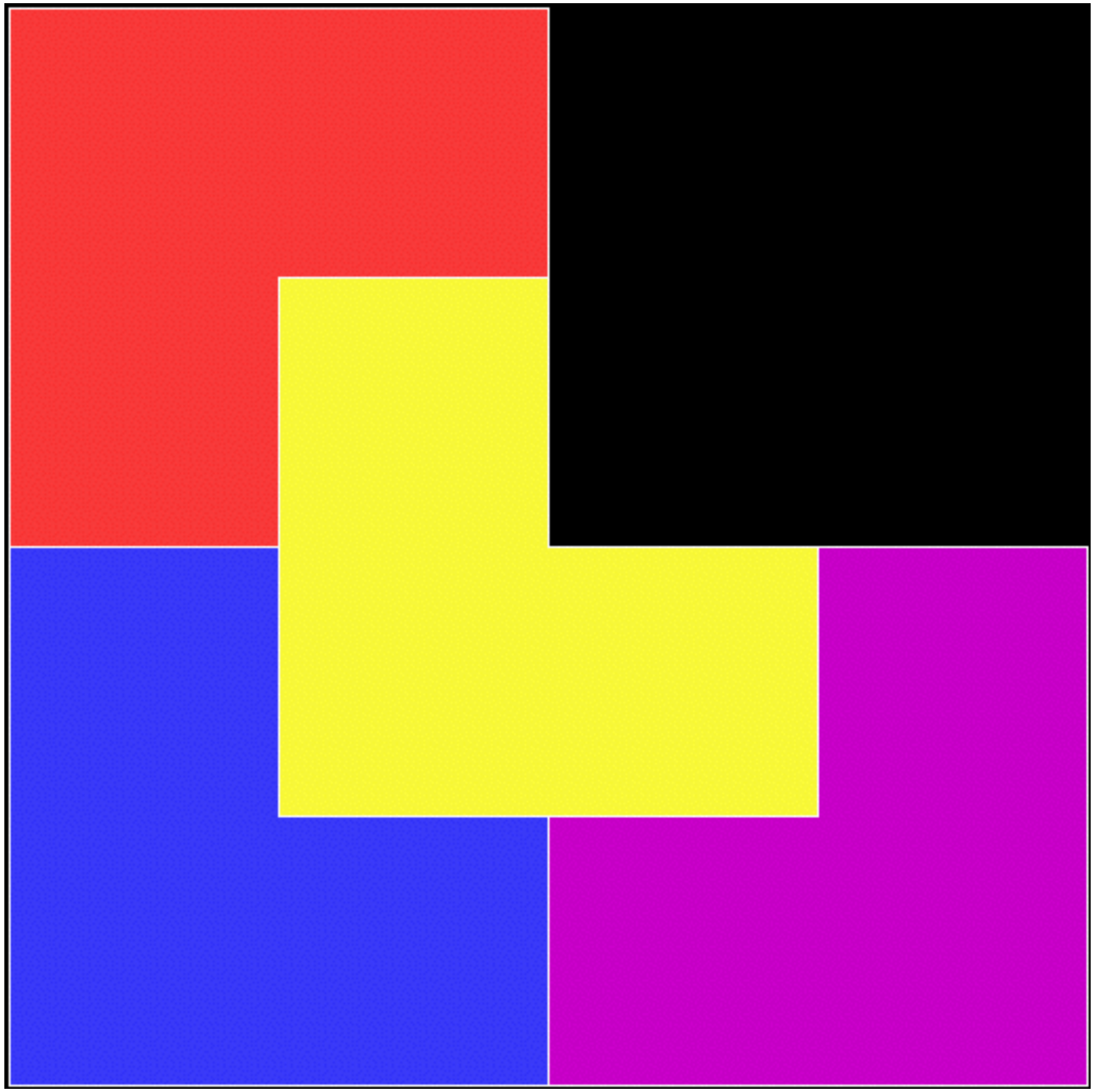
Sphinx tiling





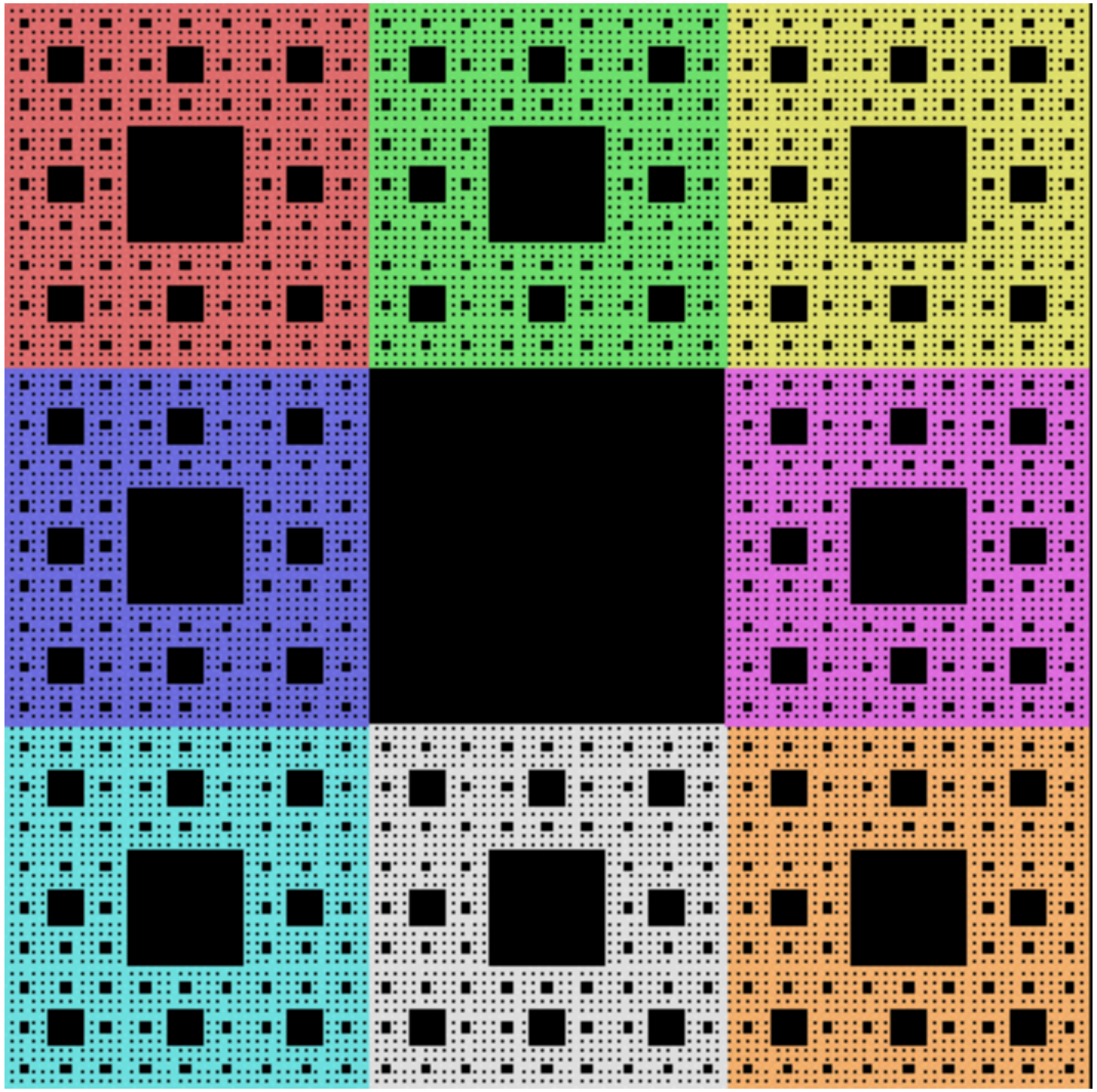
Rep-tile

Rep-tile to fractal



Rep-tile

Sierpinski carpet

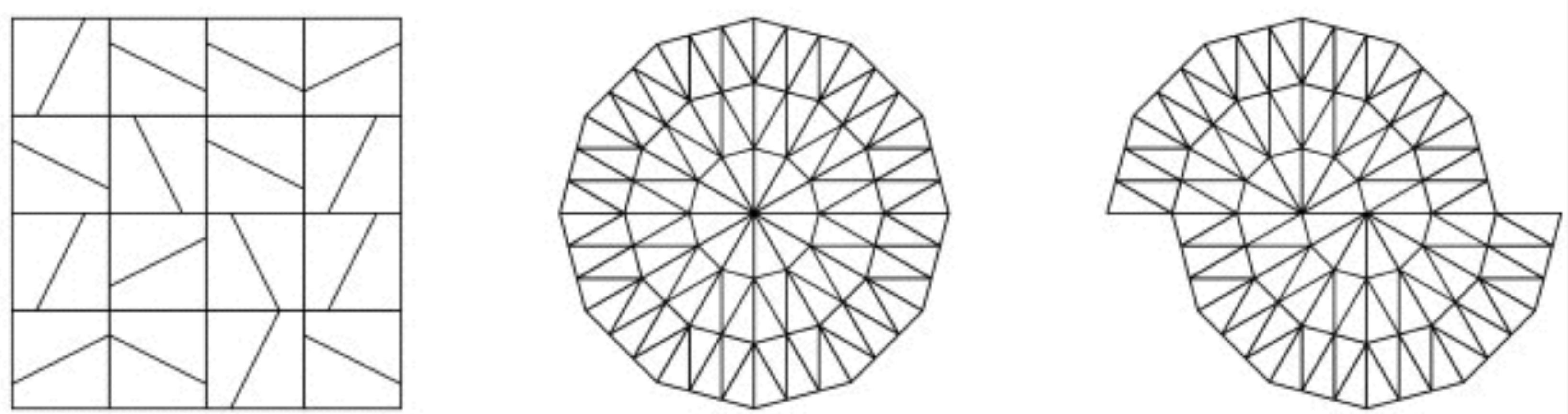


Aperiodic Tiling

Aperiodic Tiling

Aperiodic: No (full) translation symmetry

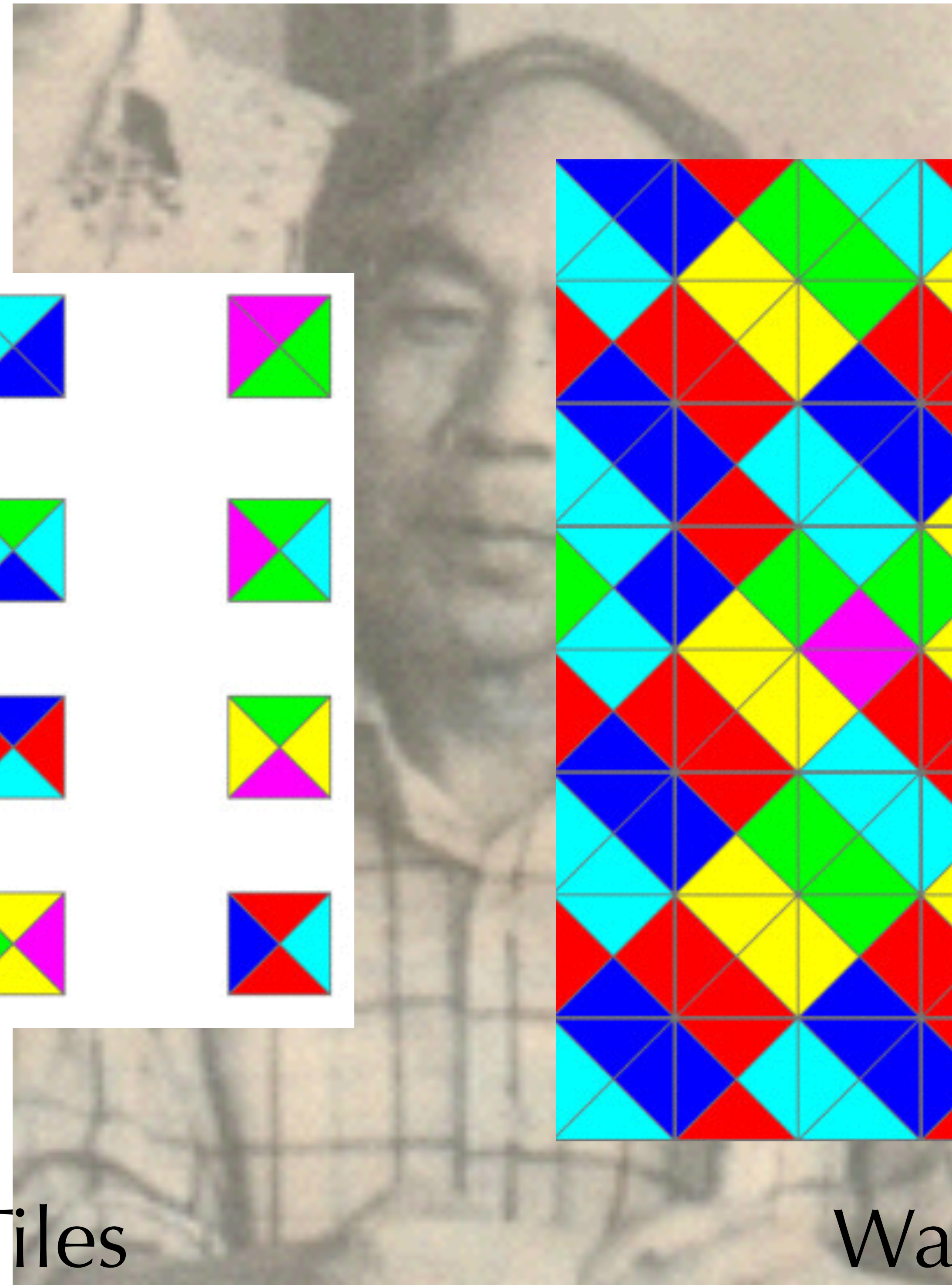
Model for quasicrystals, discovered by Dan Shechtman in 1982 (Nobel Prize 2011)

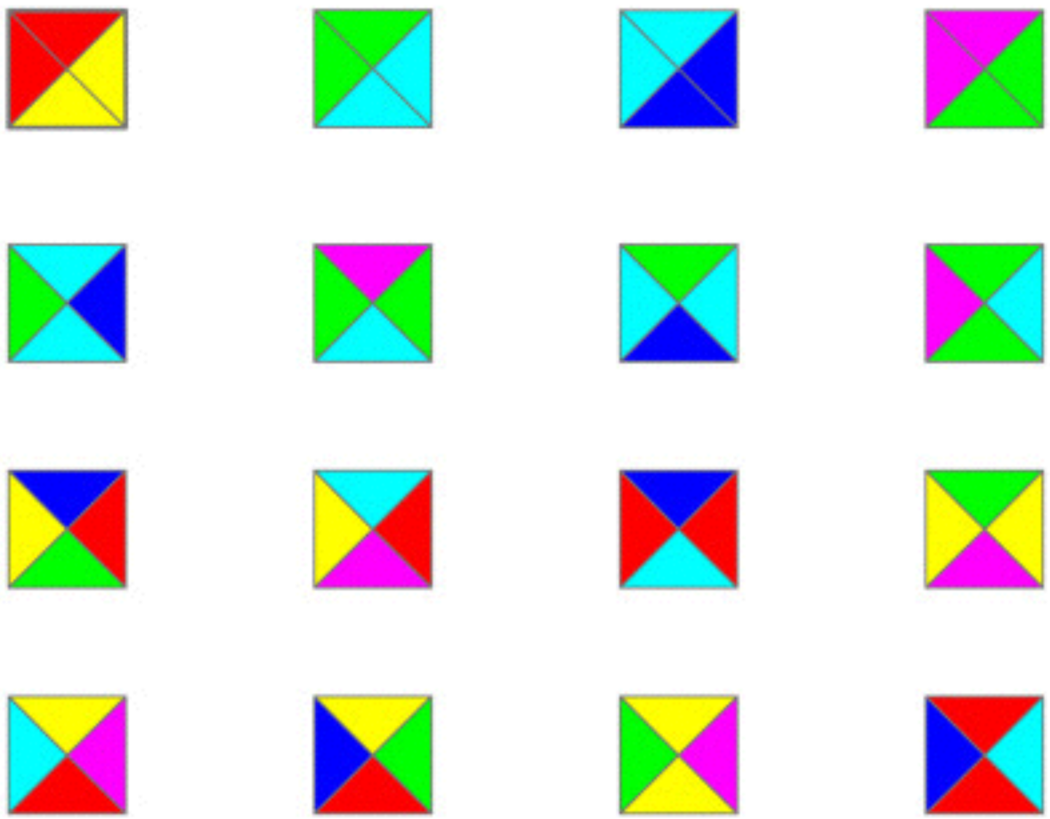


Although the tilings above are aperiodic, the same tile can make periodic tiling

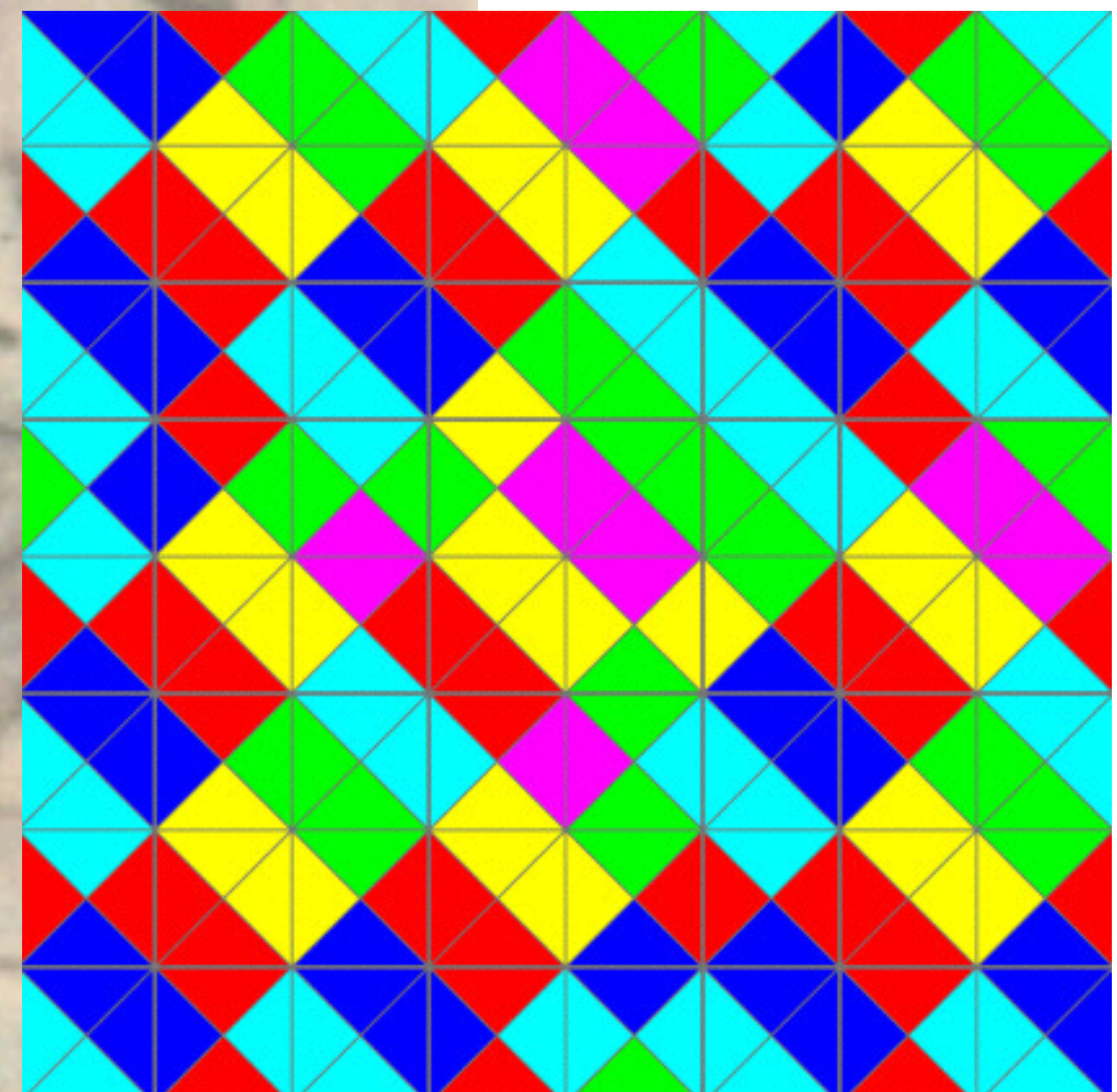
Are there aperiodic tilings, such that the tiles cannot make periodic tiling?

Wang Hao 王浩 (1921/5/20 - 1995/5/13)





16 Wang Tiles



Wang Tiling

Wang Hao 王浩

1961 Wang Hao: If a set of Wang tiles can tile a plane but not periodically, then the tiling problem for the Wang tiles is not decidable

1966 Robert Berger: a set of 20426 Wang tiles, can have tiling but has no periodic tiling

1964 Robert Berger: PhD thesis, example with 104 tiles

1968 Donald Knuth: 92 tiles

1966 Hans Läuchli (cited by 1975 article by Wang Hao): 40 tiles

1967 Raphael Robinson (unpublished): 52 tiles

1969 Raphael Robinson (published 1971): 56 tiles

1977 Raphael Robinson: 24 tiles

1978 Robert Ammann: 16 tiles

1996 Jarkko Kari: 6 colors, 14 tiles

1998 Karel Culik: 5 colors, 13 tiles

2015/6/23 Emmanuel Jeandel, Michael Rao: 4 colors, 11 tiles, minimum

An aperiodic set of 11 Wang tiles

Emmanuel Jeandel and Michael Rao

June 23, 2015

Abstract

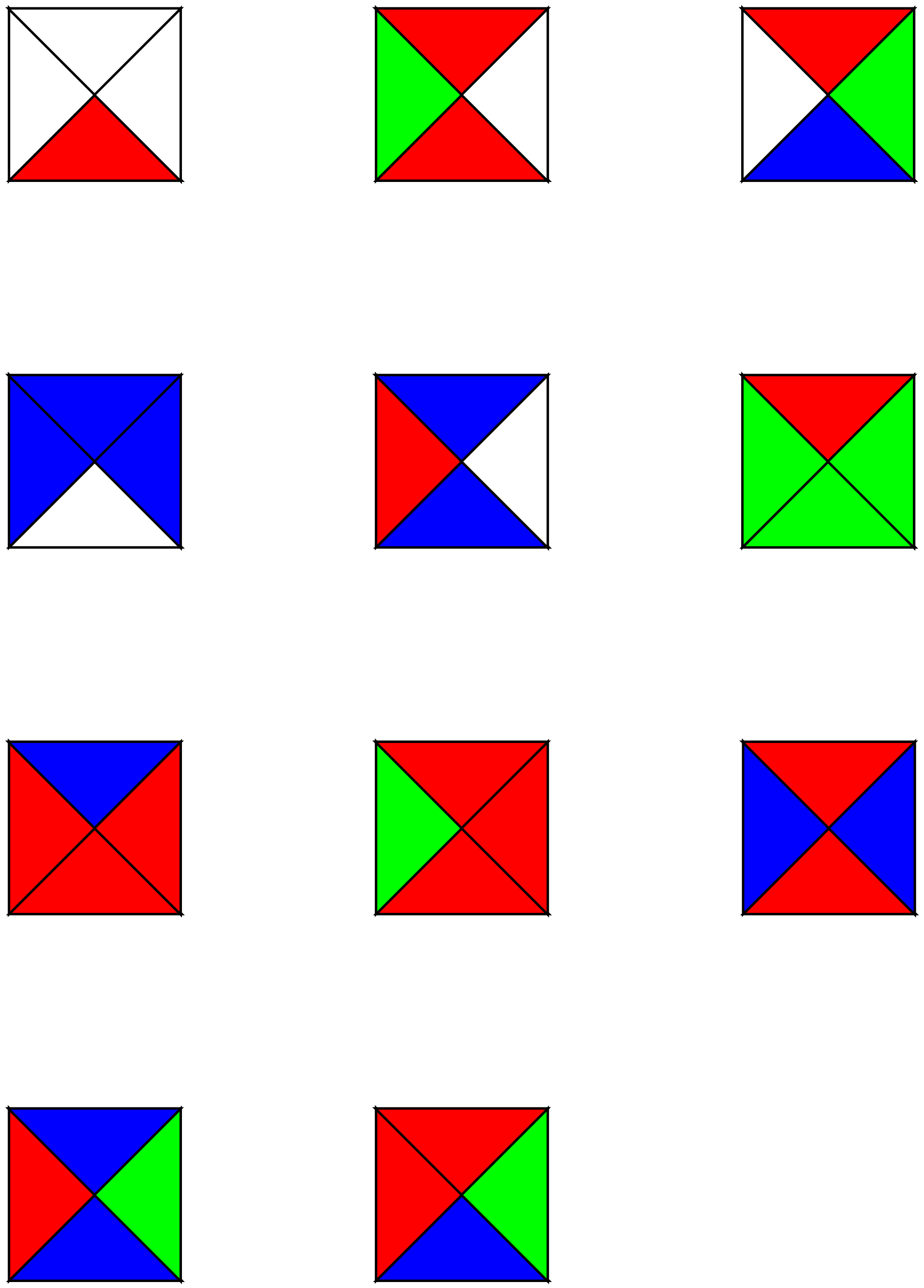
A new aperiodic tile set containing 11 Wang tiles on 4 colors is presented. This tile set is minimal in the sense that no Wang set with less than 11 tiles is aperiodic, and no Wang set with less than 4 colors is aperiodic.

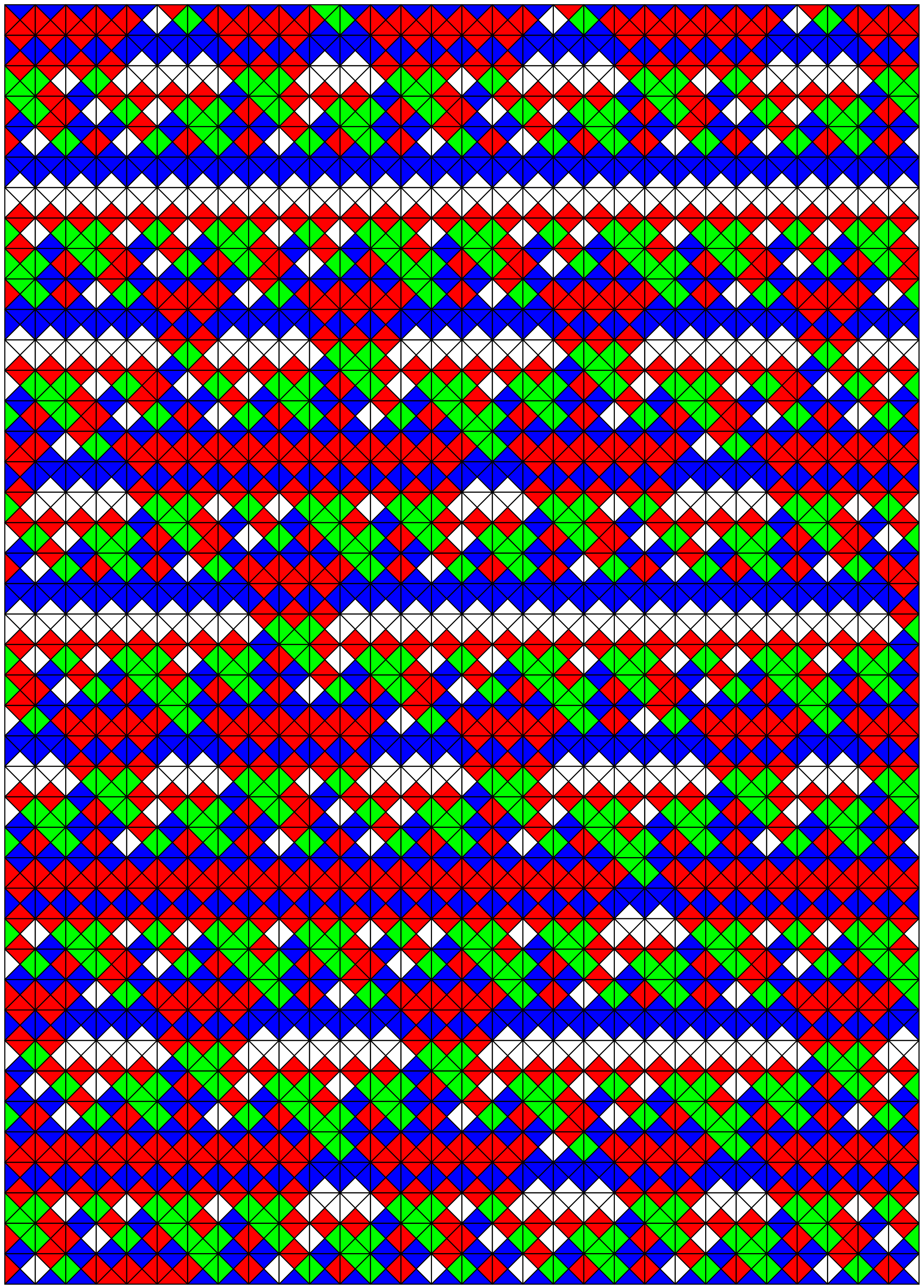
Wang tiles are square tiles with colored edges. A tiling of the plane by Wang tiles consists in putting a Wang tile in each cell of the grid $\mathbb{Z}$ so that contiguous edges share the same color. The formalism of Wang tiles was introduced by Wang [Wan61] to study decision procedures for a specific fragment of logic (see section 1.1 for details).

Wang asked the question of the existence of an aperiodic tile set: A set of Wang tiles which tiles the plane but cannot do so periodically. His student Berger quickly gave an example of such a tile set, with a tremendous number of tiles. The number of tiles needed for an aperiodic tileset was reduced during the years, first by Berger himself, then by others, to obtain in 1996 the previous record of an aperiodic set of 13 Wang tiles. (see section 1.2 for an overview of previous aperiodic sets of Wang tiles).

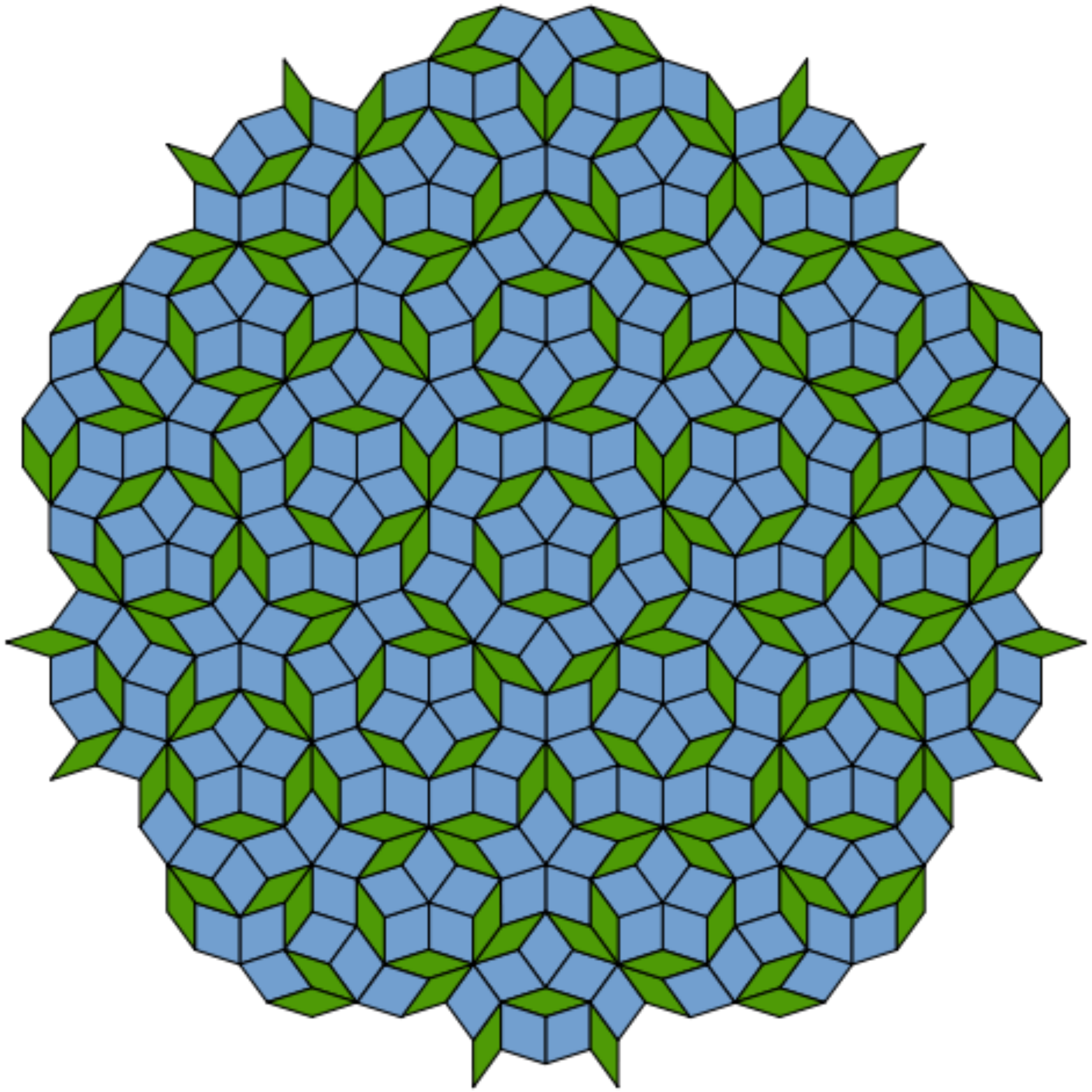
While reducing the number of tiles may seem like a tedious exercise in itself,

Wang Hao 王浩

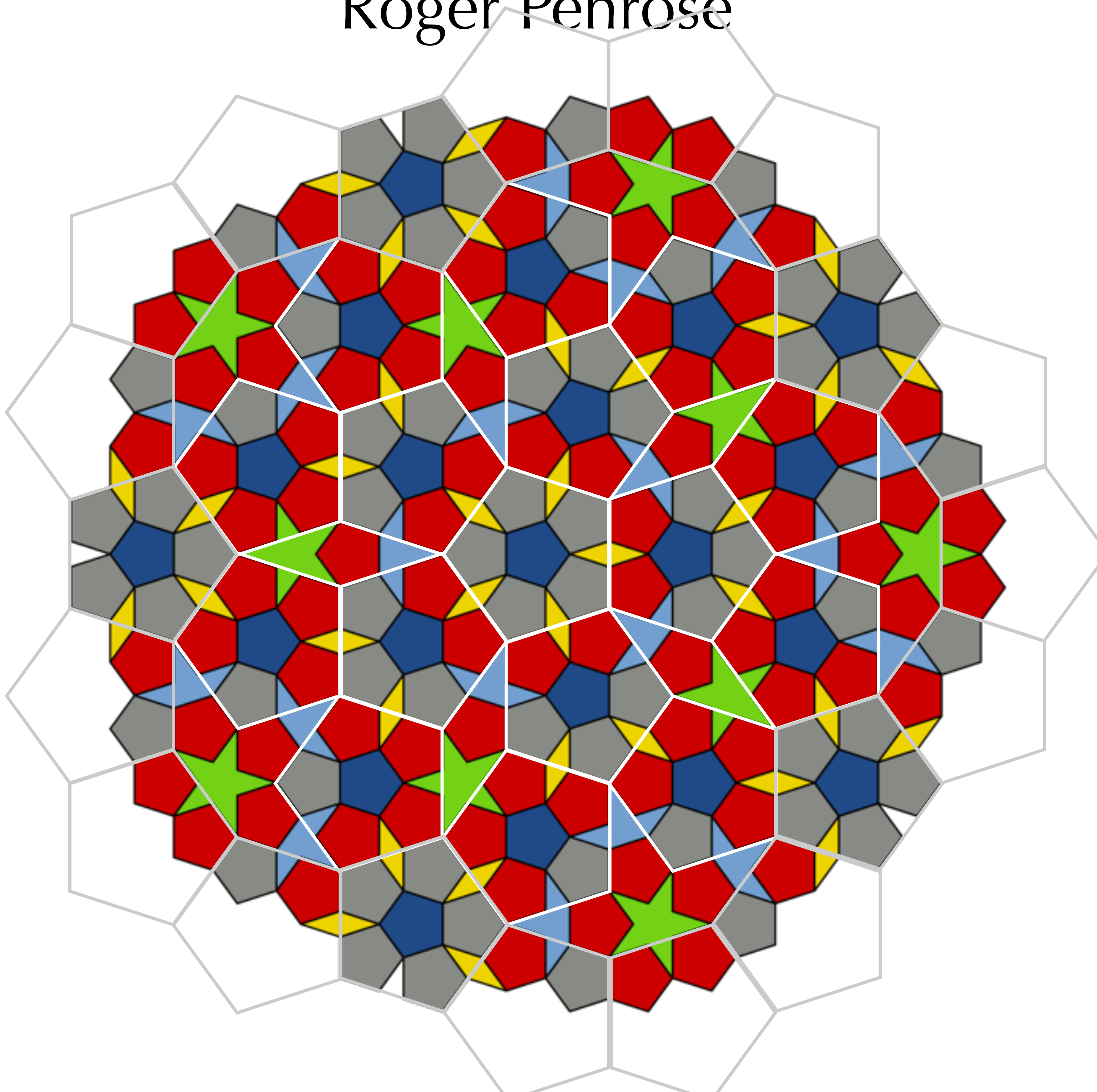




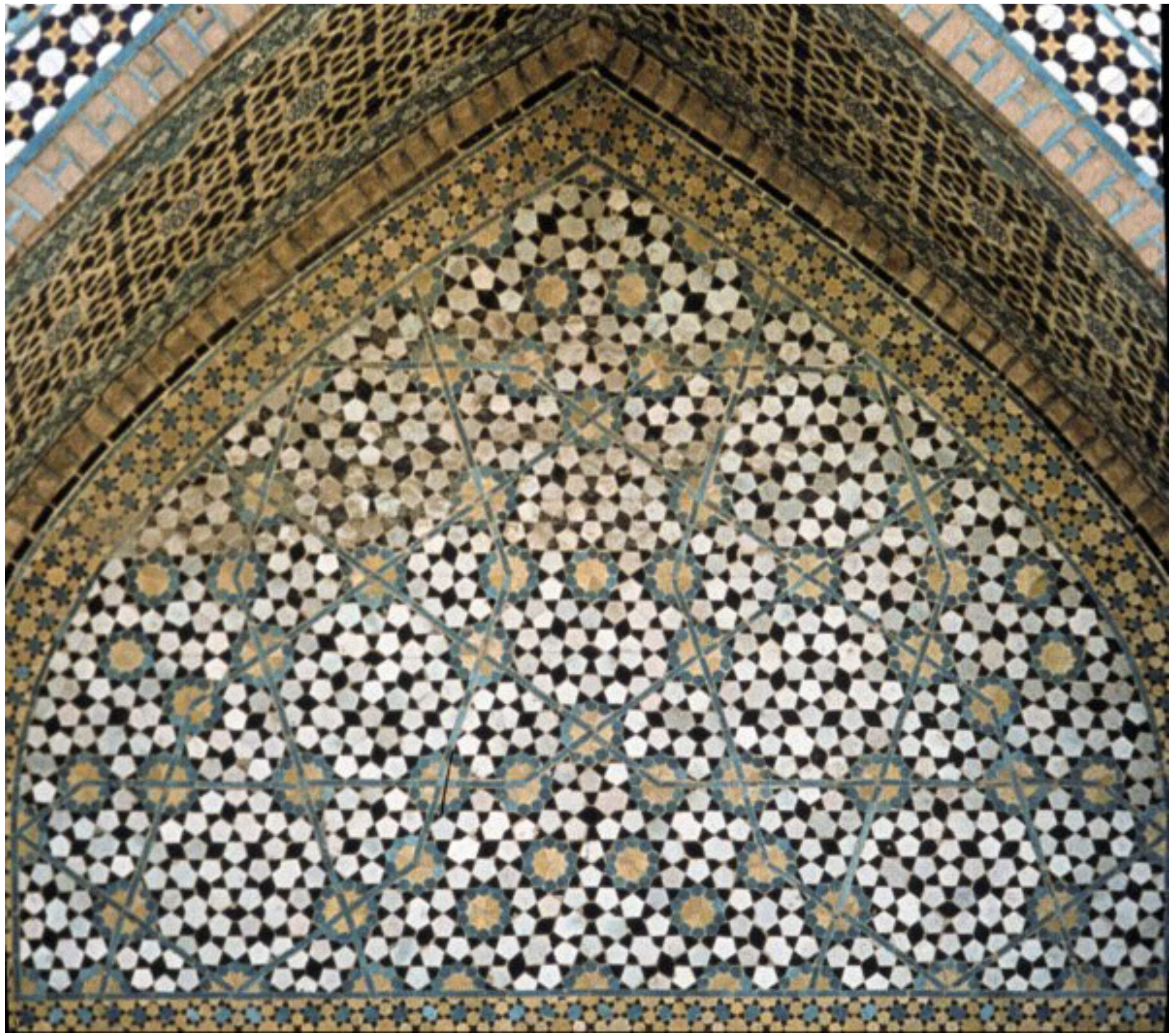
Roger Penrose



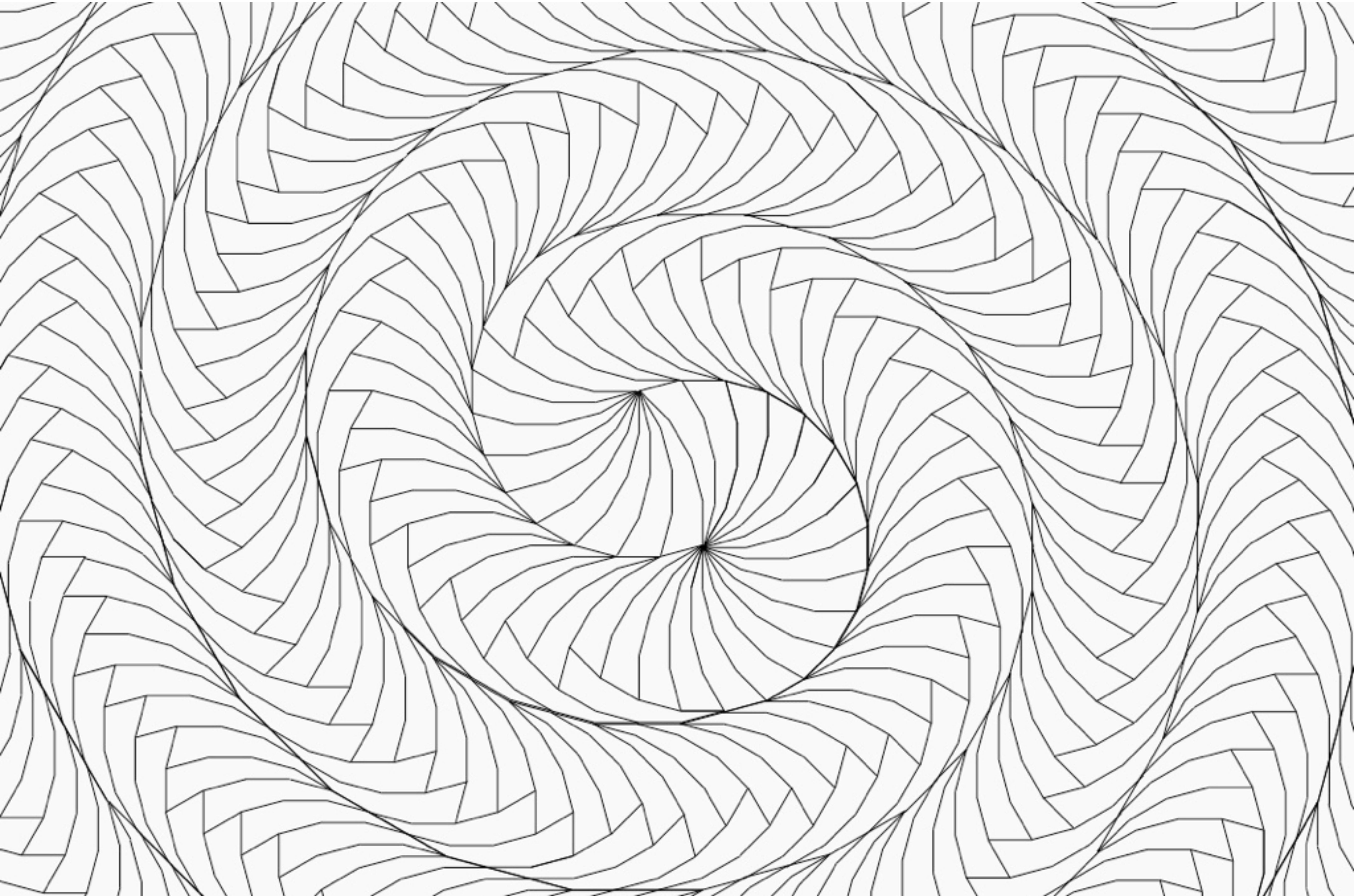
Roger Penrose



Darb-i Imam, Iran, 1453



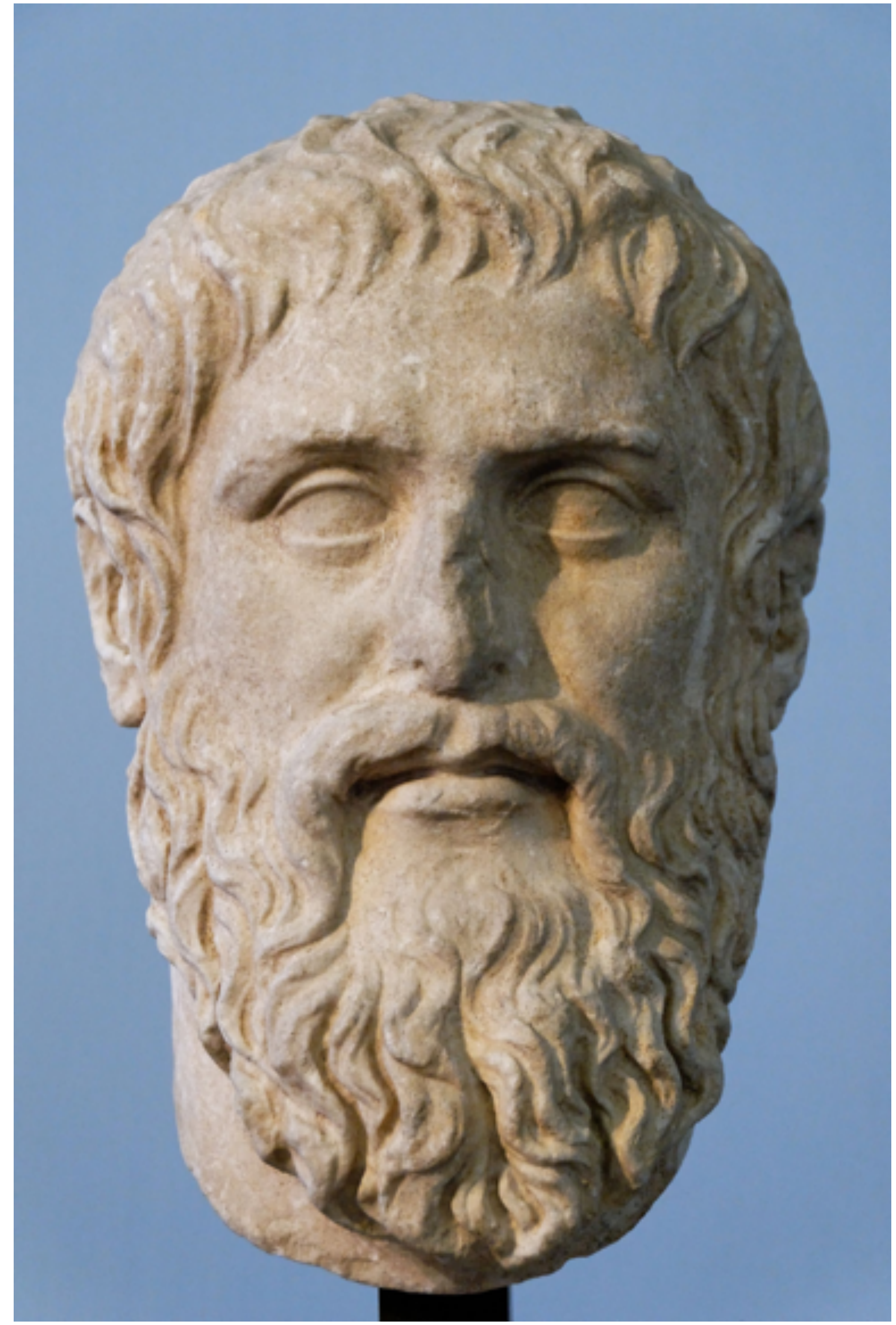
Spiral Tiling



Solids with Regular Polygon Faces

Plato (42?BC - 34?BC)

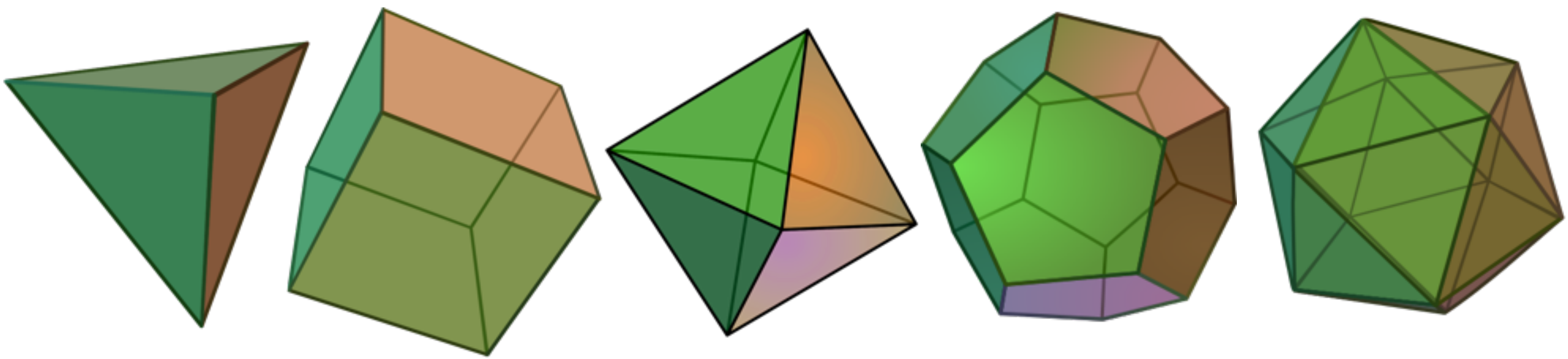
Philosopher
Mathematician



Most Perfect: Platonic Solids

All faces are the same regular polygon, and all vertices are the same

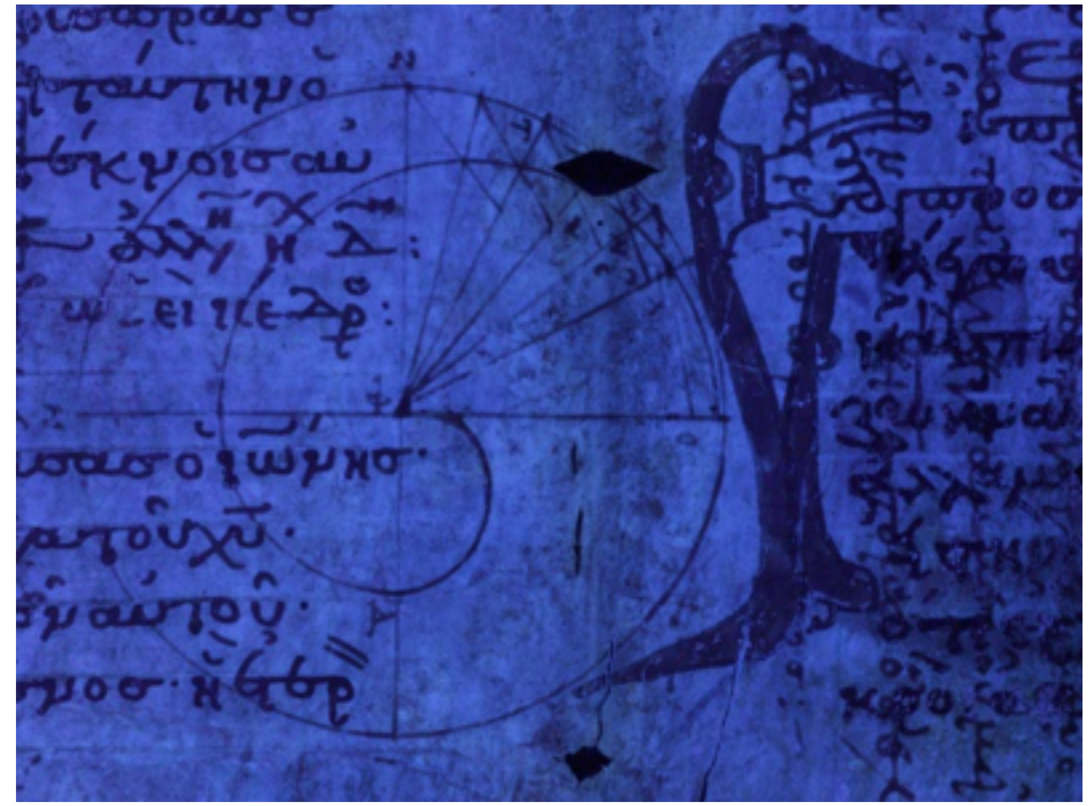
Theaetetus (417 – 369 BC) proved that there are precisely five regular convex polyhedra

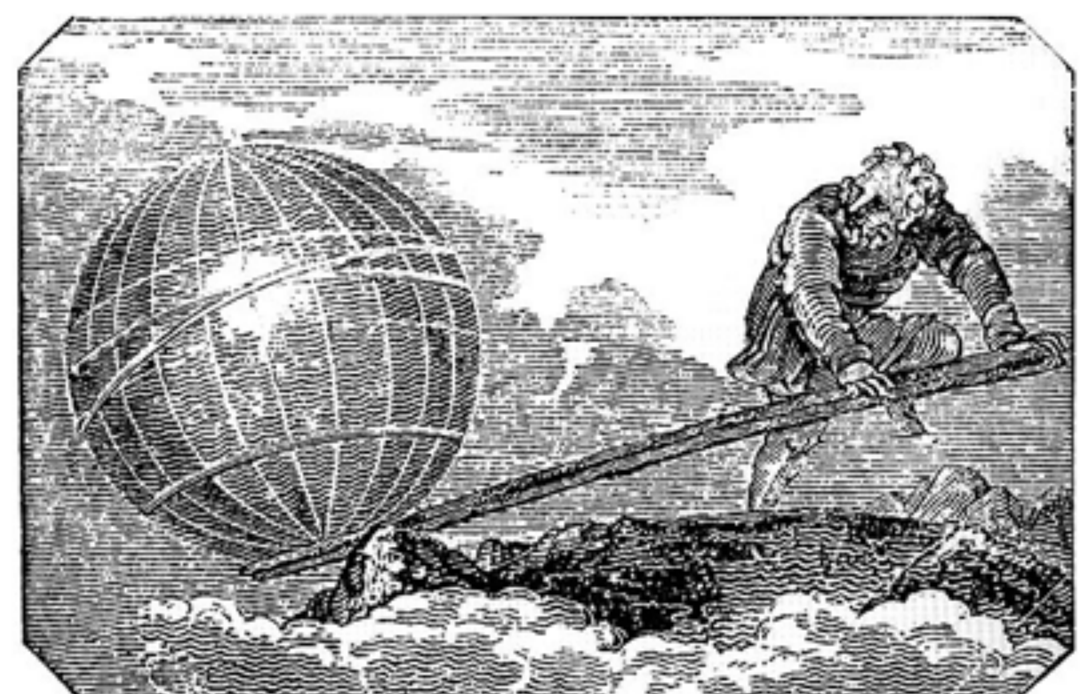


tetrahedron cube octahedron dodecahedron icosahedron

Archimedes (287BC-212BC)

Mathematician
Physicist
Engineer
Astronomer





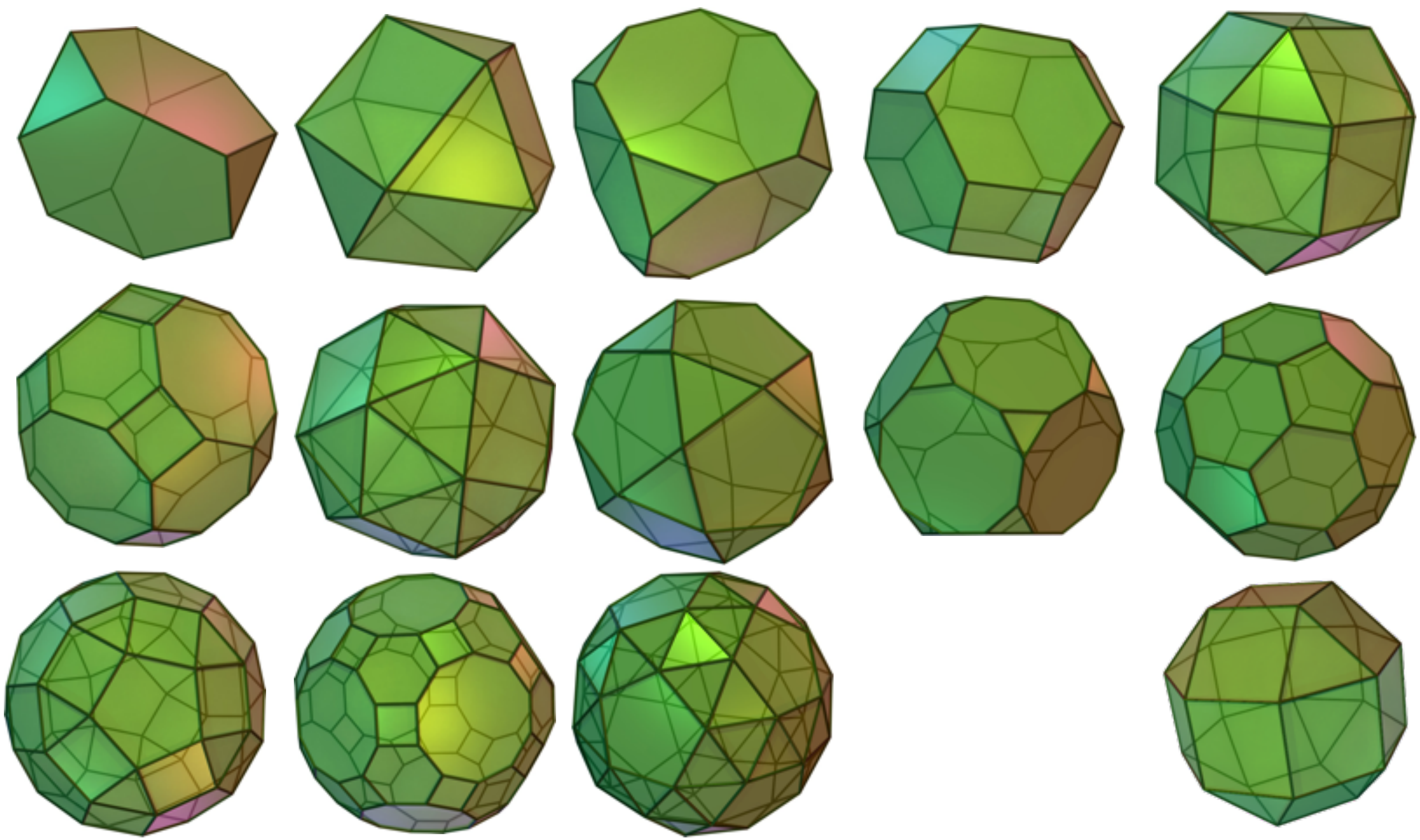
Perfect: Archimedean Solids

All faces are regular polygons, and all vertices are the same

Pappus refers to a lost work of Archimedes that listed 13 polyhedra

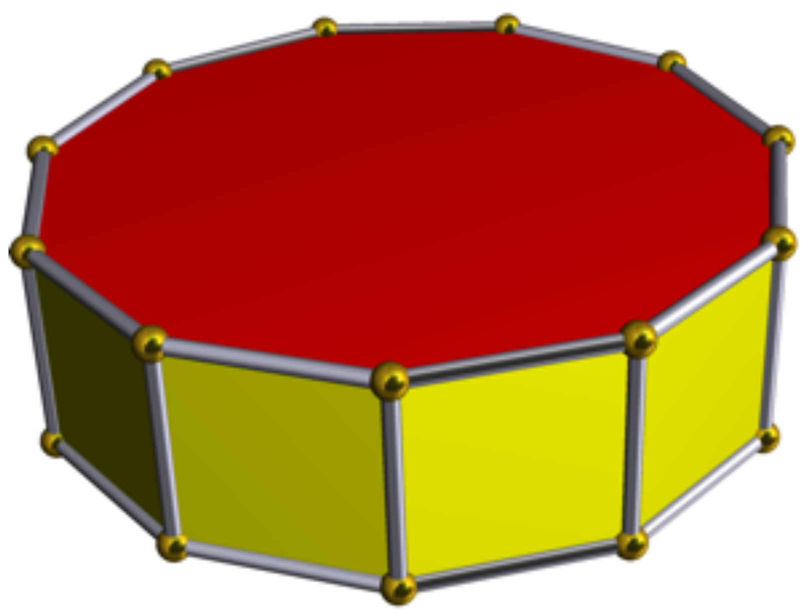
1620 Johannes Kepler rediscovered 13 (but stated 14 somewhere)

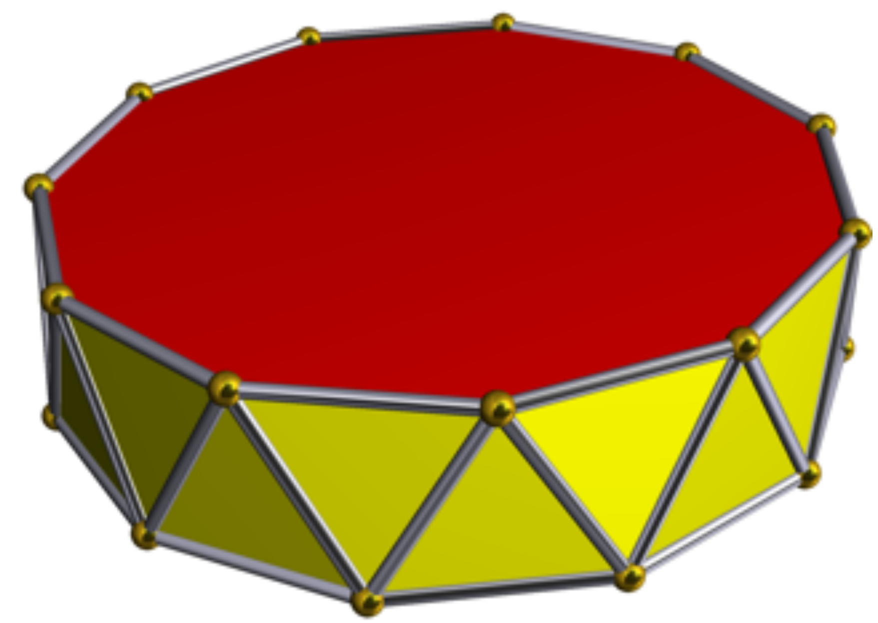
1905 Duncan Sommerville discovered pseudo-rhombicuboctahedron



Perfect: Prism and Antiprism

Theorem: Strictly convex solid, with regular polygon faces, and all vertices are the same ⇒ Platonic solid, Archimedean solid, prism, or antiprism



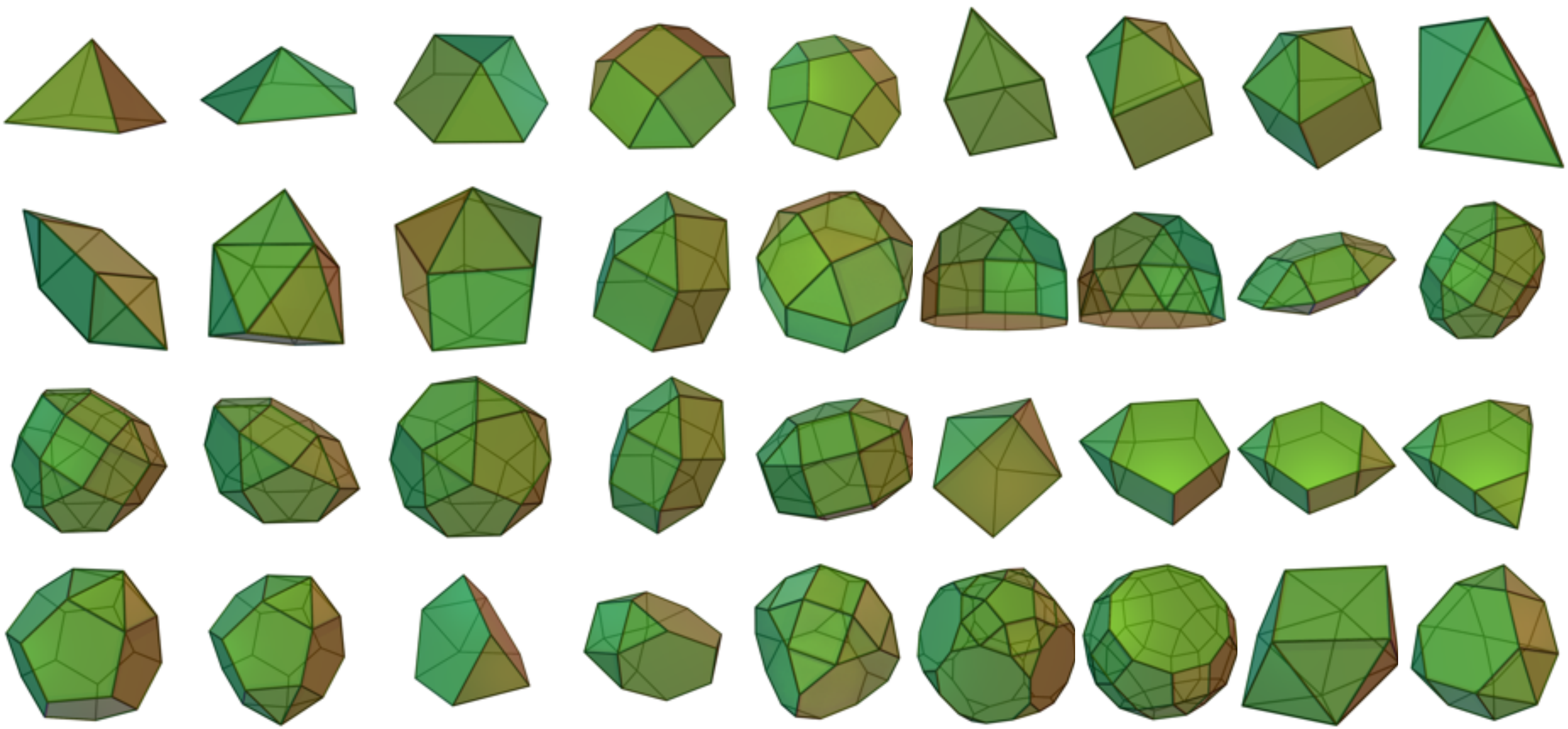


Imperfect: Johnson Solid

Strictly convex solid, with regular polygon face. Vertices no need to be the same

1966 Johnson listed 92

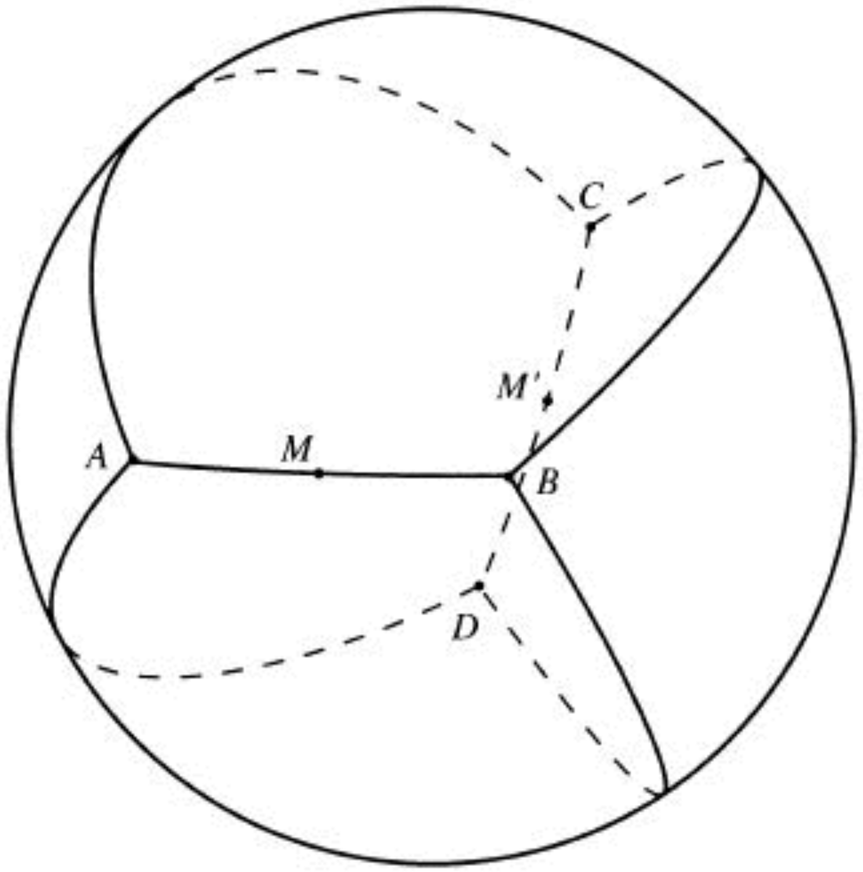
1969 Zalgaller proved the list is complete

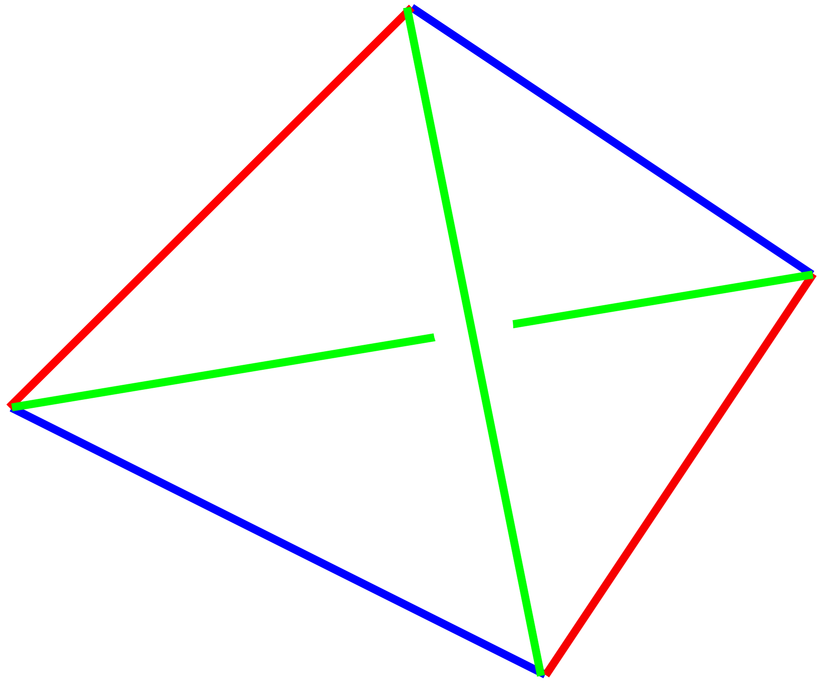


Monohedral Tiling of Sphere

Tiling of Sphere by Congruent Triangle

4 x △ = 🌏 ⇔ sum of three angles = 2π



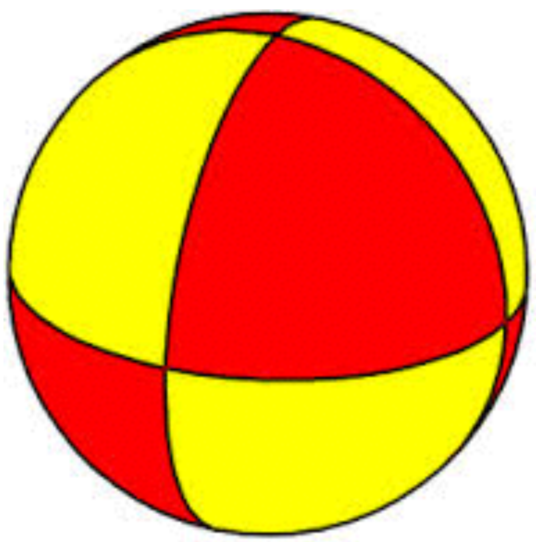
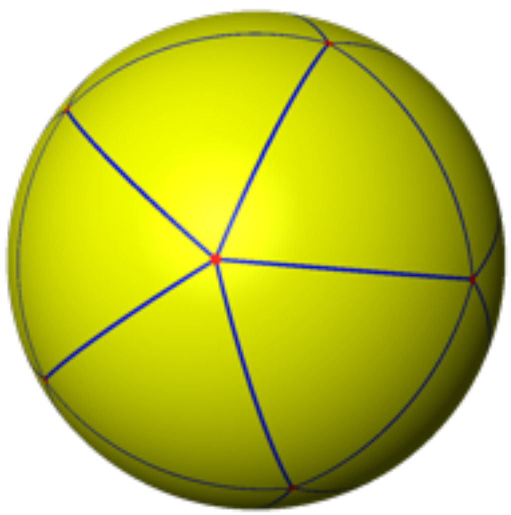
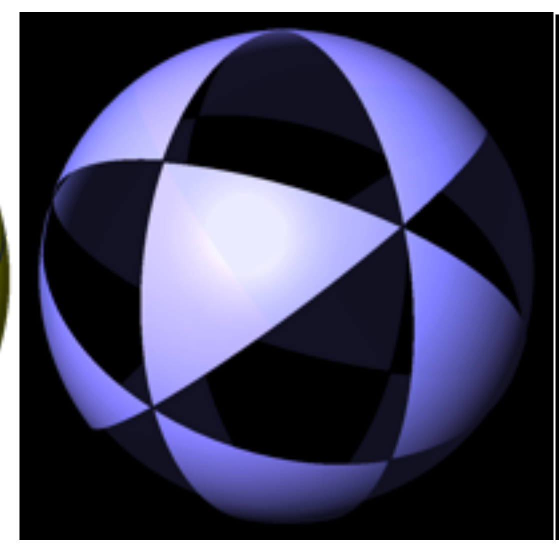
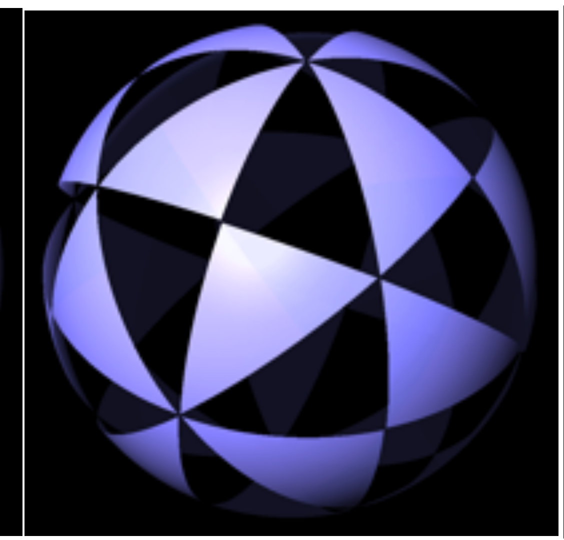
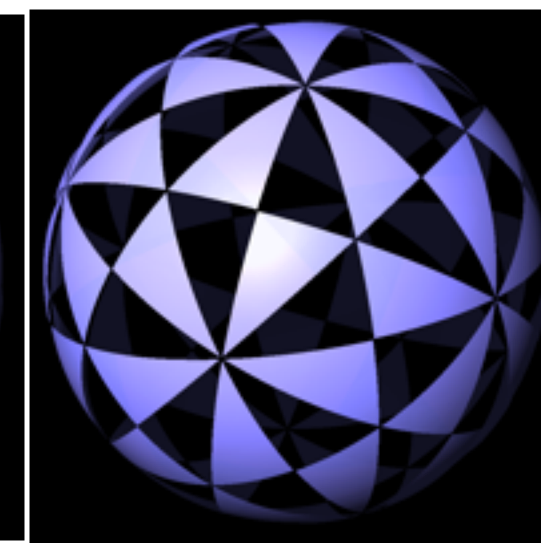


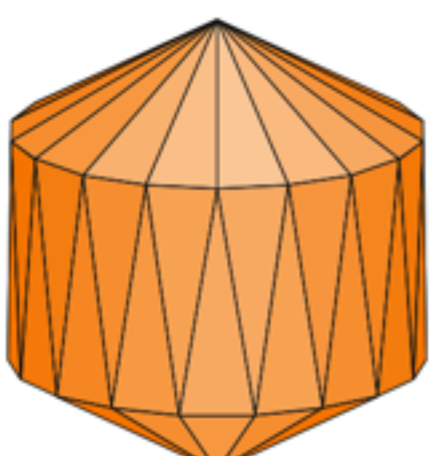
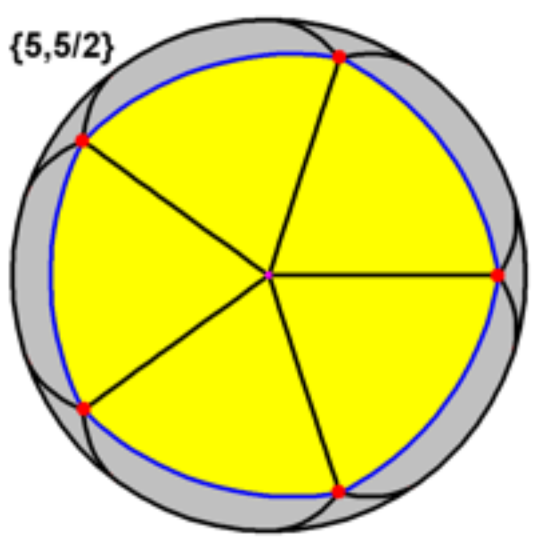
Tiling of Sphere by Congruent Triangle

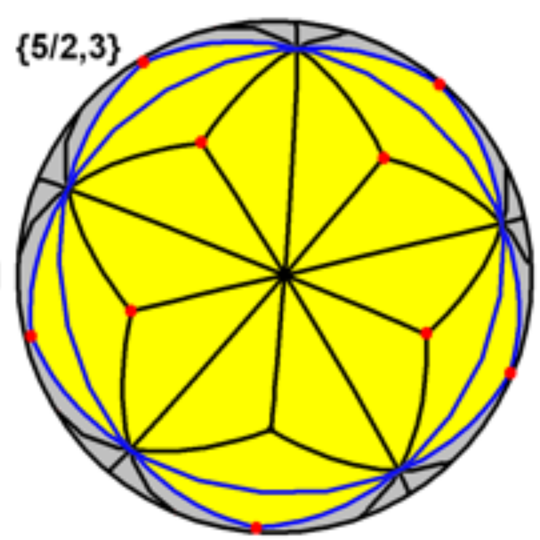
1922 Sommerville studied isosceles triangle

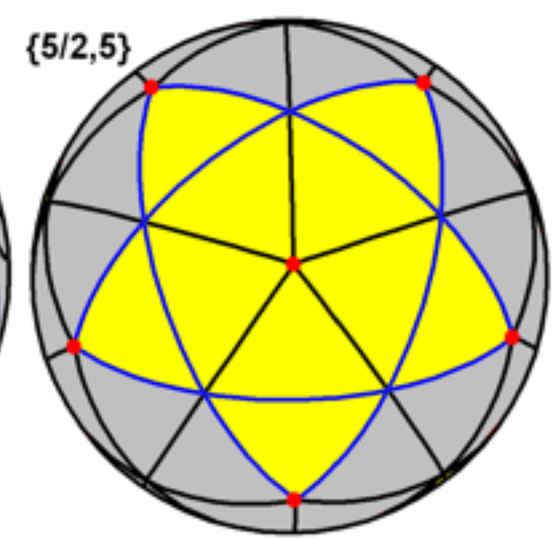
2002 Ueno and Agaoka complete classification, about 20 edge-to-edge families

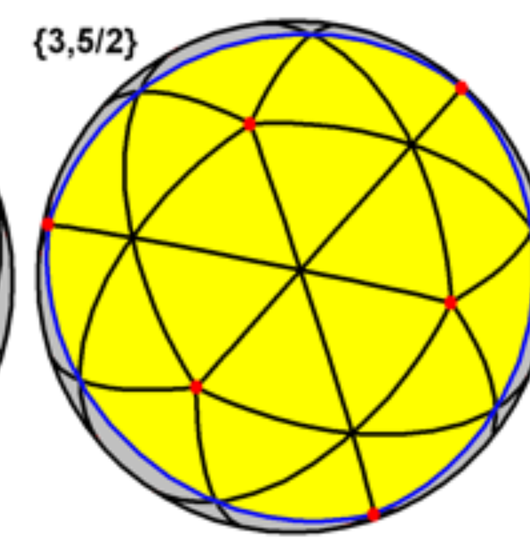
Many non-edge-to-edge examples



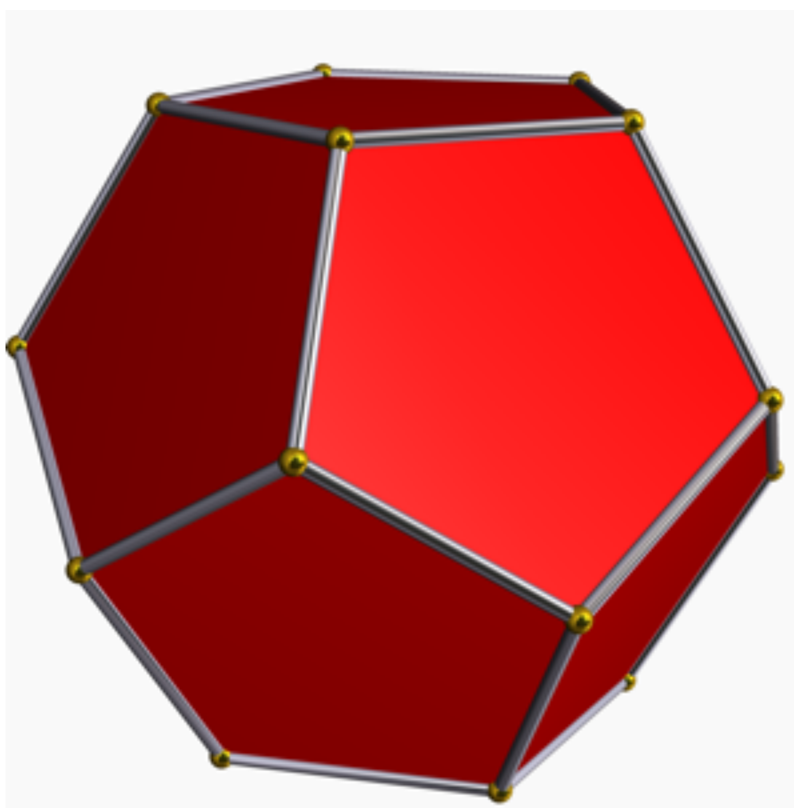


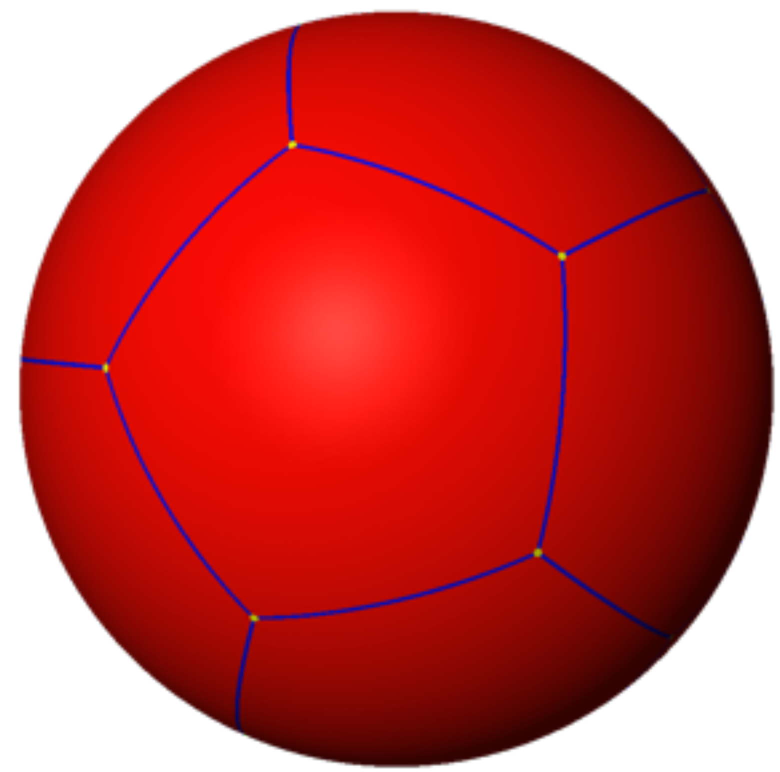


Tiling of Sphere by Congruent Pentagon

Edge-to-edge tiling of the sphere by congruent n-gons ⇒ $n = 3, 4, 5$

After $n = 3$, next “extreme” case is $n = 5$. Basic example: dodecahedron





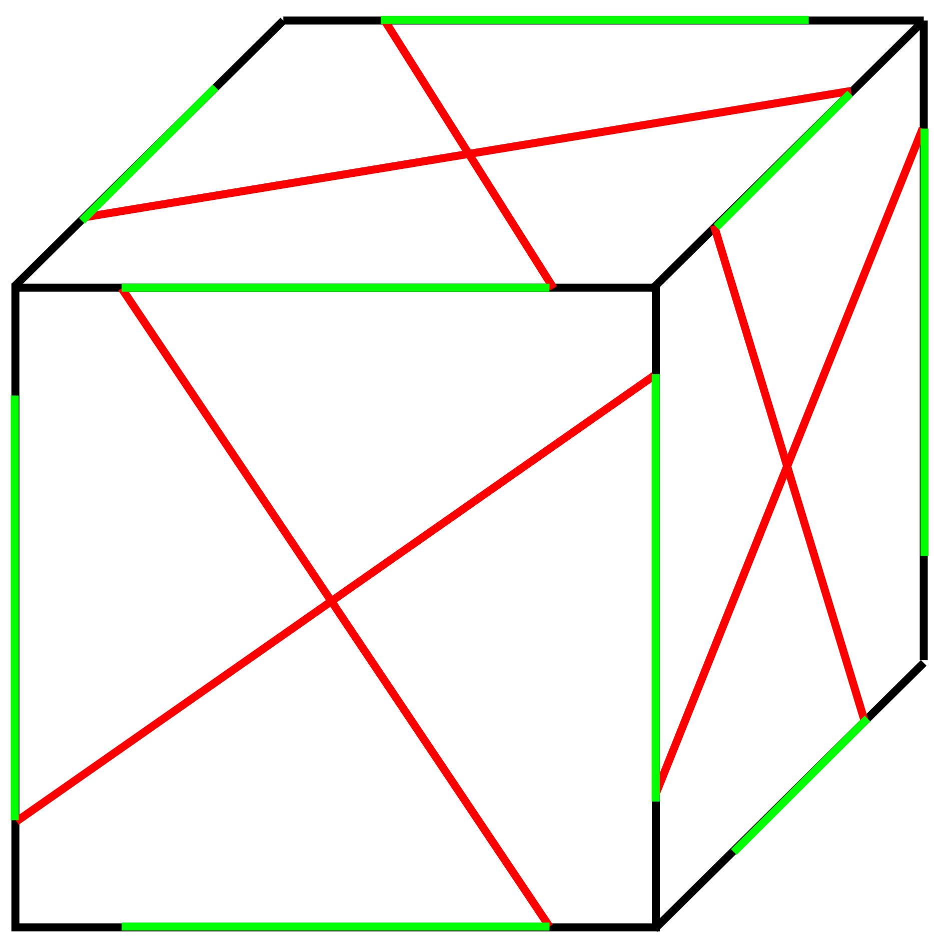
First Construction

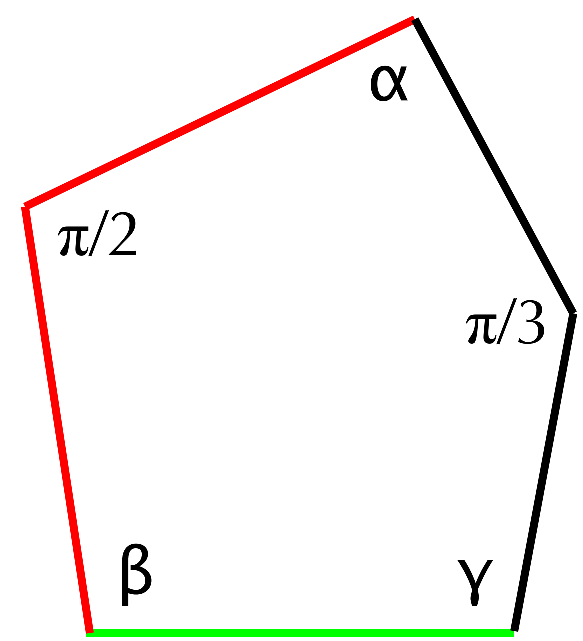
Pentagonal subdivision of Platonic solids

Tetrahedron ⇒ 12 ⬠ = ⚽

Cube & Octahedron ⇒ 24 ⬠ = ⚽

Dodecahedron & Icosahedron ⇒ 60 ⬠ = ⚽



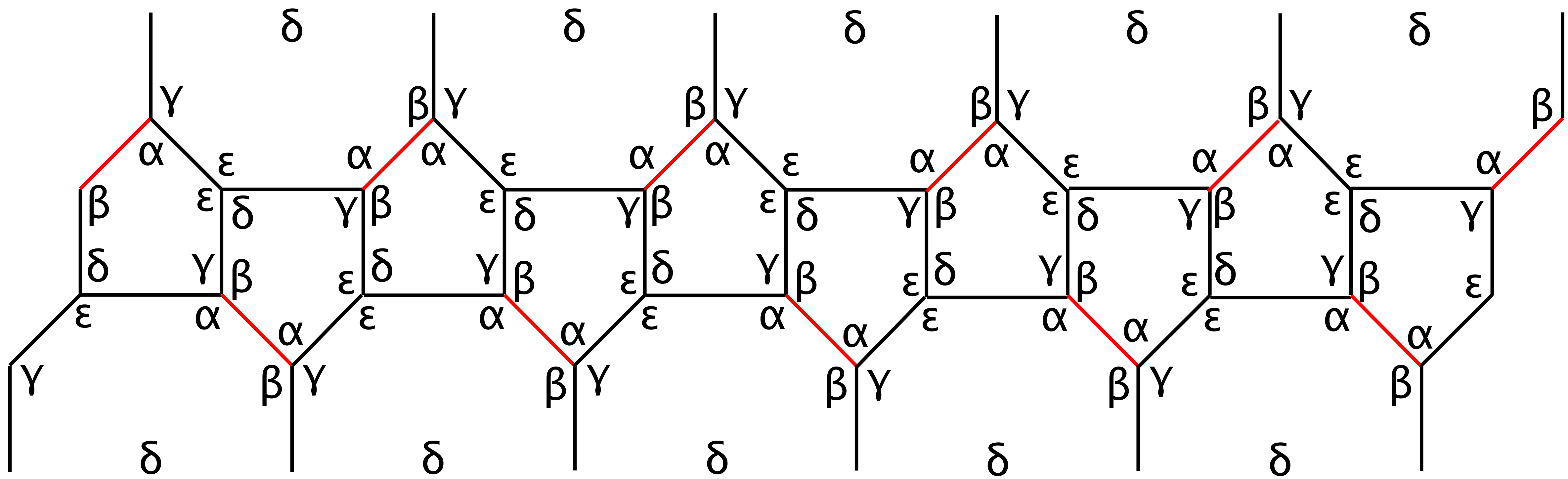


$\alpha + \beta + \gamma = 2\pi$

Second Construction

Earth map tiling: $\alpha + \beta + \gamma = 2\pi$, $\delta = 2\pi/n$

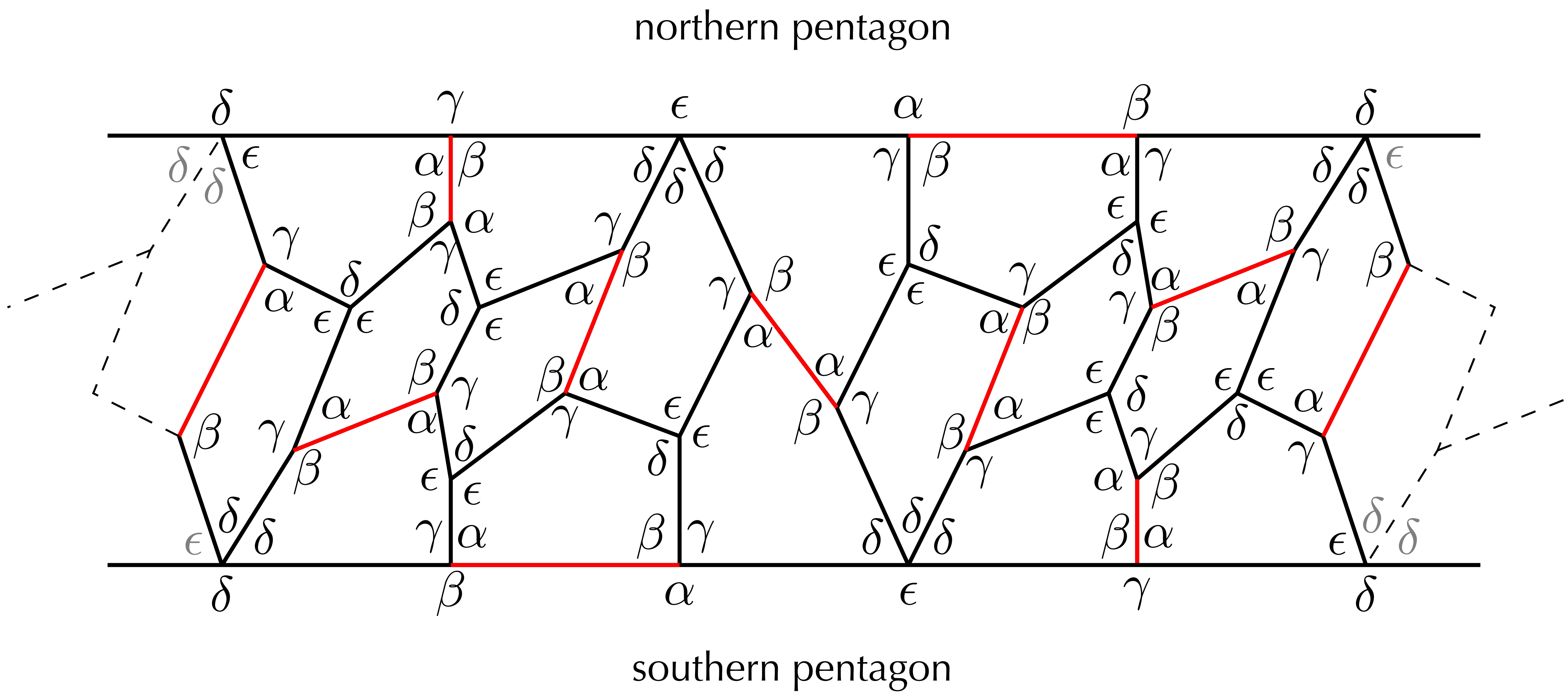
north pole



south pole

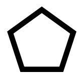
Third Construction

$\alpha + \beta + \gamma = 2\pi$, $\delta = 2\pi/5$, $\varepsilon = 4\pi/5$



Tiling of Sphere by Congruent Pentagon

Gao-Shi-Yan

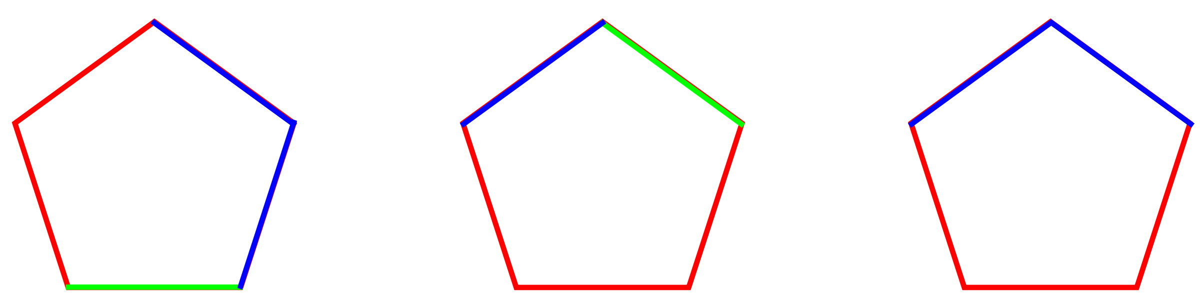
12  =  ⇒ deformed dodecahedron (first construction)

Akama-Yan

8 tilings by congruent equilateral pentagon: 3 pentagonal subdivisions, 4 earth map tilings (n = 4, 5, 6, 6), and 1 special tiling

Cheuk-Cheung-Yan

For >12 pentagons, no tiling with edge length combo a^2b^2c, a^3bc, a^3b^2



Thank You